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Bounding Betti Numbers of Sets Definable in O-Minimal Structures Over the Reals

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Bounding Betti Numbers of Sets Definable in O-Minimal Structures Over the Reals

submitted by

Mahana Clutha

for the degree of Doctor of Philosophy

of the

University of Bath

Department of Computer Science

January 2011

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Summary

A bound for Betti numbers of sets definable in o-minimal structures is presented.

An axiomatic complexity measure is defined, allowing various concrete complexity measures for definable functions to be covered. This includes common concrete measures such as the degree of polynomials, and complexity of Pfaffian functions.

A generalisation of the Thom-Milnor Bound [17, 19] for sets defined by the conjunction of equations and non-strict inequalities is presented, in the new context of sets definable in o-minimal structures using the axiomatic complexity measure. Next bounds are produced for sets defined by Boolean combinations of equations and inequalities, through firstly considering sets defined by sign conditions, then using this to produce results for closed sets, and then making use of a construction to approximate any set defined by a Boolean combination of equations and inequalities by a closed set.

Lastly, existing results [12] for sets defined using quantifiers on an open or closed set are generalised, using a construction from Gabrielov and Vorobjov [11] to approximate any set by a compact set. This results in a method to find a general bound for any set definable in an o-minimal structure in terms of the axiomatic complexity measure. As a consequence for the first time an upper bound for sub-Pfaffian sets defined by arbitrary formulae with quantifiers is given. This bound is singly exponential if the number of quantifier alternations is fixed.

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For Isobel, Jane and Florence. Three generations of exceptional women.

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Chapter 1

Preliminaries

1.1 Introduction

This thesis considers the problem of relating the complexity of a formula to the complexity of the “shape” it describes. The natural questions arising from this initial statement of the problem are “what type of formula?”, “how do we measure the complexity of the formula?” and “how do we measure the complexity of the shape?”.

There are a variety of answers to the first two questions. Existing work in this area mostly deals with polynomial functions, with degree as complexity measure. Some work has been done with Pfaffian functions (for example [22]), see Section 1.5.2 for definitions and the relevant complexity measure. This thesis, however, considers a more general type of formula, that is sets definable in o-minimal structures over the reals (see Section 1.5.1). There is not standard method of measuring the complexity of such formula, hence one of the main contributions of this work is to define an axiomatic complexity measure to solve this problem.

The final question has a standard answer – we measure the complexity of “shapes”, i.e. some sort of geometric complexity, using Betti numbers. These are defined in Section 1.5.6. In the following we frequently bound the sum of the Betti numbers of a given set, which obviously bounds each individual Betti number (as they are a finite sequence of integers).

1.2 Previous Work

In 1964 Milnor [17] considered a real algebraic set $S \subset \mathbb{R}^n$, defined by polynomial equations

$$f_1(x_1, \dots, x_m) = \dots = f_p(x_1, \dots, x_m) = 0$$

of maximum degree d . He showed the sum of the Betti numbers of S is bounded by $d(2d-1)^{n-1}$. Milnor also showed that if the p equations above were instead non-strict inequalities, and they have total degree $d' = \deg(f_1) + \dots + \deg(f_p)$, then the Betti sum of the set defined by these inequalities is bounded by $\frac{1}{2}(2+d')(1+d')^{n-1}$. Various alternate versions of the above results

have been published (see Petrovskii and Oleinik [18], Thom [19]), and modified, simpler proofs offered (for example Basu, Pollack, Roy [2]).

In 1999 Basu [1] showed that if a closed semi-algebraic set $S \subset \mathbb{R}^n$ is defined as the intersection of a real algebraic set $Q = 0$, where Q is a polynomial with $\deg(Q) \leq d$, whose real dimension is n' , with a set defined by a quantifier-free Boolean formula with no negations with atoms of the form $P_i = 0$, $P_i \geq 0$, $P_i \leq 0$, for $1 \leq i \leq s$, and the degree of polynomials P_i being at most d , then the sum of the Betti numbers of S is bounded by $s^{n'}(O(d))^n$. In 2004 Basu, Pollack and Roy [3] refined this bound to

$$\sum_{i=0}^{n'} \sum_{j=0}^{n'-i} \binom{s}{j} 6^j d(2d-1)^{n-1}.$$

In the same paper a bound is also constructed for the sum of the i -th Betti numbers over the realisation of all sign conditions of a set of polynomials on an algebraic set.

The preceding results all concern sets defined in terms of polynomials. In 1999, Zell [22] produced a similar result for semi-Pfaffian sets, directly following the above methods. Further to this, a survey of upper bounds on sets defined by Pfaffian and Noetherian functions is presented in [9].

In 2005, Gabrielov and Vorobjov [10] described a way to replace any arbitrary semi-algebraic set, defined by a arbitrary Boolean quantifier-free formula, by a compact set with coinciding Betti numbers, and presented a bound of $O(k^2 d)^n$. In the second edition of [2] an alternative, simpler proof of this result is given, showing homotopy equivalence rather than simply coinciding Betti numbers.

As the Tarski-Seidenberg Theorem (see for example [7]) states that in the semi-algebraic case, every first-order formula is equivalent to a quantifier-free formula, the polynomial case is essentially complete, save refinements of bounds. Sets defined in o-minimal structures however, do not generally have this property, for example see [8].

Gabrielov, Vorobjov and Zell [12] showed how to associate a spectral sequence with a surjective map, and used this to find an upper bound for open and closed sets defined by first-order formulae. In particular, results are presented for sub-Pfaffian sets defined by first-order formulae. This is one of the main ingredients of the last chapter of this thesis, and indeed one of the primary aims of this work is to remove the restriction that requires sets to be open or closed.

In 2009, Gabrielov and Vorobjov [11] presented a method for approximating any o-minimal set S by a compact o-minimal set T , with each Betti number of S being bounded by that of T , and, in a particular case, these sets are shown to be homotopy equivalent. This result leads to a refinement of some of the above bounds in the semi-algebraic and semi- and sub- Pfaffian cases, and, more importantly, presents a method that can be used for any o-minimal set.

1.3 Outline

Existing work suggests two areas for expansion.

Firstly, we seek to reproduce the above results, this time for functions defined in o-minimal structures instead of simply polynomials or Pfaffian functions. We start with sets defined by equalities, then closed sets, then sets defined by an arbitrary Boolean combination, before finally considering sets defined using quantifiers.

The second new area derives naturally from this problem – established complexity measures for polynomials (for example degree, number of monomials, additive complexity) and Pfaffian functions exist, but there is no conventional way to measure the complexity of a function defined in an o-minimal structure. So, before proceeding with the above calculations, we create an axiomatic complexity definition. The definition presented in this work was inspired by [4].

The remainder of this chapter consists of definitions and results from existing work that are used in our proofs, and concludes with a presentation of our new axiomatic complexity definition. In Chapter 2 we start with the case of a set defined by multiple equalities, following the methods of Milnor, then move on to sets defined by the conjunction of equations and non-strict inequalities. Chapter 3 initially shows how to bound the Betti numbers of sign conditions, then uses this to bound the Betti numbers of closed sets. Lastly a construction is presented to approximate a set defined by any Boolean combination of equations and inequalities by a closed set, and this, in conjunction with the previous result, gives a bound of the Betti numbers of the original set. The final chapter considers sets defined using quantifiers, and is the most significant contribution of this thesis. In [12] a method is presented to bound the Betti numbers of a surjective map, using a particular spectral sequence, and then this is used in the case of the projective map to bound the Betti numbers of a open or closed set defined with quantifiers. In [11], a method is shown to approximate any set with a compact set. The main result of this thesis combines these two results, along with the preceding work, to give a bound for any set defined in an o-minimal structure using quantifiers. As a consequence for the first time an upper bound for sub-Pfaffian sets defined by arbitrary formulae with quantifiers is given. This bound is singly exponential if the number of quantifier alternations is fixed.

Our methods are necessary for sub-Pfaffian sets, as there is no equivalent to Tarski-Seidenberg for polynomials. As is made clear in Section 1.5.2, a restricted sub-Pfaffian set need not be semi-Pfaffian, and there is no quantifier elimination. This is therefore also true for sets definable in o-minimal structures over the reals (as this includes sets defined by Pfaffian functions).

1.4 Main Results

This thesis has three main contributions to knowledge:

- A new axiomatic complexity metric is defined in Section 1.6, which improves on the presentation in [4].
- Known quantitative bounds in polynomial and Pfaffian settings are generalized to our new setting of sets definable in o-minimal structures. These new results reduce to existing results in the polynomial and Pfaffian cases.

- New results are produced in the o-minimal setting for quantified formula over sets which need not be open or closed. The application of this to the Pfaffian case is also new. Main Result A (Theorem 4.9.8) gives a bound in terms of the summation of sets, and Main Result B (Theorem 4.10.7, really a Corollary of Theorem 4.9.8) gives a specific bound in terms of the complexity metric.

1.5 Background

1.5.1 O-Minimal Structures

A very good text on this subject is [20]. The following two definitions are taken directly from [6]:

Definition 1.5.1. A *structure expanding the real closed field R* is a collection $S = (S_n)_{n \in \mathbb{N}}$, where each S_n is a set of subsets of the affine space R^n , satisfying the following axioms:

1. All algebraic subsets of R^n are in S_n
2. For every n , S_n is a Boolean subalgebra of the powerset of R^n
3. If $A \in S_m$, and $B \in S_n$, then $A \times B \in S_{m+n}$
4. If $p : R^{n+1} \rightarrow R^n$ is the projection on the first n coordinates, and $A \in S_{n+1}$, then $p(A) \in S_n$.

The elements of S_n are called the *definable subsets of R^n* . The structure S is said to be *o-minimal* if, moreover, it satisfies:

5. The elements of S_1 are precisely the finite unions of points and intervals.

Definition 1.5.2. A map $f : A \rightarrow R^p$ (where $A \subset R^n$) is called *definable* if its graph is a definable subset of $R^n \times R^p$.

Conic Structure

Let $\overline{B}_n(a, r)$ denote the closed ball in \mathbb{R}^n with radius r , centered at a , and let $S_n(a, r)$ denote the sphere in \mathbb{R}^n with radius r , centered at a . The following Theorems are taken from [6]

Theorem 1.5.3 (Local Conic Structure). *Let $A \subset \mathbb{R}^n$ be a closed definable set, and a a point in A . There is a $r > 0$ such that there exists a definable homeomorphism h from the cone with vertex a and base $S_n(a, r) \cap A$ onto $\overline{B}_n(a, r) \cap A$ satisfying $h|_{S_n(a, r) \cap A} = \text{Id}$ and $|h(x) - a| = |x - a|$ for all x in the cone.*

The following theorem tells us that any closed definable set is homotopy equivalent to the intersection of that set with a ball of sufficiently large radius. This is useful later when we have a closed set defined by a combination of equations and/or inequalities, and we require a compact set – we only need to add one additional inequality.

Theorem 1.5.4 (Conic Structure at Infinity). *Let $A \subset \mathbb{R}^n$ be a closed definable set. Then there exists $r \in \mathbb{R}$, $r > 0$, such that for every r' , $r' \geq r$, there is a definable deformation retraction from A to $A_{r'} = A \cap \overline{B_n}(0, r')$ and a definable deformation retraction from $A_{r'}$ to A_r .*

Proof. Let us suppose A is not bounded. Through an inversion map $\phi : \mathbb{R}^n \setminus 0 \rightarrow \mathbb{R}^n \setminus 0$, $\phi(x) = x/|x|^2$, we can reduce to the property of local conic structure for $\phi(A) \cup 0$ at 0. \square

This theorem can be visualised as follows: consider a simplicial complex K equivalent to A . Take r large enough so that each simplex in K intersects with the ball of radius r , and each bounded simplex is a subset of the ball. Then each unbounded simplex can be “shrunk” in, and be replaced with an equivalent bounded simplex, without affecting the homotopy type.

1.5.2 Pfaffian Functions

We present a definition for Pfaffian functions from [9], modified to only include the real case:

Definition 1.5.5. A *Pfaffian chain* of the order $r \geq 0$ and degree $\alpha \geq 1$ in an open domain $G \subset \mathbb{R}^n$ is a sequence of analytic functions f_1, \dots, f_r in G satisfying differential equations

$$df_j(\mathbf{x}) = \sum_{1 \leq i \leq n} g_{ij}(\mathbf{x}, f_1(\mathbf{x}), \dots, f_j(\mathbf{x})) dx_i$$

for $1 \leq j \leq r$. Here $g_{ij}(\mathbf{x}, y_1, \dots, y_j)$ are polynomials in $\mathbf{x} = (x_1, \dots, x_n)$, y_1, \dots, y_j of degrees not exceeding α . A function $f(\mathbf{x}) = P(\mathbf{x}, f_1(\mathbf{x}), \dots, f_j(\mathbf{x}))$, where $P(\mathbf{x}, f_1(\mathbf{x}), \dots, f_j(\mathbf{x}))$ is a polynomial of a degree not exceeding $\beta \geq 1$, is called a *Pfaffian function* of order r and degree (α, β) . Note that the Pfaffian function f is defined only in the domain G where all functions f_1, \dots, f_r are analytic, even if f itself can be extended as an analytic function to a larger domain.

We present a few examples to illustrate, taken from [9].

Example 1.5.6. (a) *Pfaffian functions of order 0 and degree $(1, \beta)$ are polynomials of degree not exceeding β .*

(b) *The exponential function $f(x) = e^{ax}$ is a Pfaffian function of order 1 and degree $(1, 1)$ in \mathbb{R} , due to the equation $df(x) = af(x)dx$.*

(c) *The function $f(x) = 1/x$ is a Pfaffian function of order 1 and degree $(2, 1)$ in the domain $\{x \in \mathbb{R} : x \neq 0\}$, due to the equation $df(x) = -f^2(x)dx$.*

(d) *The polynomial $f(x) = x^m$ can be viewed as a Pfaffian function of order 2 and degree $(2, 1)$ in the domain $\{x \in \mathbb{R} : x \neq 0\}$ (but not in \mathbb{R}), due to the equations $df(x) = mf(x)g(x)dx$ and $dg(x) = -g^2(x)dx$, where $g(x) = 1/x$.*

The expansion of \mathbb{R} by sets defined by Pfaffian functions was proven to be o-minimal in [21]. The following lemmas and definitions are also taken from [9].

Lemma 1.5.7. *The sum (resp. product) of two Pfaffian functions f_1 and f_2 of orders r_1 and r_2 and degrees (α_1, β_1) and (α_2, β_2) respectively, is a Pfaffian function of order $r_1 + r_2$ and degree $(\alpha, \max\{\beta_1, \beta_2\})$ (resp. $(\alpha, \beta_1 + \beta_2)$), where $\alpha = \max\{\alpha_1, \alpha_2\}$. If the two functions are defined by the same Pfaffian chain of order r , then the orders of the sum and the product are both equal to r .*

Lemma 1.5.8. *A partial derivative of a Pfaffian function of order r and degree (α, β) , is a Pfaffian function having the same Pfaffian chain of order r , and degree $(\alpha, \alpha + \beta - 1)$.*

Definition 1.5.9. A set $X \subset \mathbb{R}^n$ is called *semi-Pfaffian* in an open domain $G \subset \mathbb{R}^n$ if it consists of points in G satisfying a Boolean combination \mathcal{F} of some atomic equations and inequalities $f = 0$, $g > 0$, where f, g are Pfaffian functions having a common Pfaffian chain defined in G . We will write $X = \{\mathcal{F}\}$. A semi-Pfaffian set X is *restricted* in G if its topological closure lies in G . A semi-Pfaffian set is called *basic* if the Boolean combination is just a conjunction of equations and strict inequalities.

Definition 1.5.10. A set $X \subset \mathbb{R}^n$ is called *sub-Pfaffian* in an open domain $G \subset \mathbb{R}^n$ if it is an image of a semi-Pfaffian set under a projection into a subspace.

Definition 1.5.11. Consider the closed cube $I^{m+n} = [-1, 1]^{m+n}$ in an open domain $G \subset \mathbb{R}^{m+n}$, and the projection map $\pi : \mathbb{R}^{m+n} \rightarrow \mathbb{R}^n$. A subset $Y \subset I^n$ is called *restricted sub-Pfaffian* if $Y = \pi(X)$ for a restricted semi-Pfaffian set $X \subset I^{m+n}$.

Note that a restricted sub-Pfaffian set need not be semi-Pfaffian, see [9] for an example. This is the most significant difference between the theories of semi- and sub-Pfaffian sets on the one hand, and semialgebraic sets on the other, and is a key reason this thesis is needed.

We now present an analogue of Bezout's theorem for Pfaffian functions:

Theorem 1.5.12 (Khovanskii, [15], [16]). *Consider a system of equations $f_1 = \dots = f_n = 0$, where f_i , $1 \leq i \leq n$, are Pfaffian functions in a domain $G \subset \mathbb{R}^n$, having a common Pfaffian chain of order r , and degrees (α, β_i) respectively. Then, the number of non-degenerate solutions of this system does not exceed*

$$\mathcal{M}(n, r, \alpha, \beta_1, \dots, \beta_n) = 2^{r(r-1)/2} \beta_1 \dots \beta_n (\min\{n, r\} \alpha + \beta_1 + \dots + \beta_n - n + 1)^r.$$

1.5.3 Hardt's Triviality

The following is taken from [6].

Definition 1.5.13. Let $X \subset \mathbb{R}^m \times \mathbb{R}^n$ be a definable family, and denote by π_m the projection onto \mathbb{R}^m . Let A be a definable subset of \mathbb{R}^m , and let $X_A = X \cap (A \times \mathbb{R}^n)$. We say that the family X is *definably trivial over A* if there exists a definable set F and a definable homeomorphism $h : A \times F \rightarrow X_A$ such that, for $x \in A \times F$

$$\pi_m(h(x)) = \pi_m(x).$$

We say that h is a *definable trivialisation of X over A* . Now let Y be a definable subset of X . We say that the trivialisation h is *compatible with Y* if there is a definable subset G of F such that $h(A \times G) = Y_A$. Note that if h is compatible with Y , its restriction to Y_A is a trivialisation of Y over A .

Theorem 1.5.14 (Hardt’s Theorem for Definable Families). *Let $X \subset \mathbb{R}^m \times \mathbb{R}^n$ be a definable family. Let Y_1, \dots, Y_l be definable subsets of X . There exists a finite partition of \mathbb{R}^m into definable sets C_1, \dots, C_k such that X is definably trivial over each C_i , and moreover, the trivialisations over each C_i are compatible with Y_1, \dots, Y_l .*

This will frequently be used in a situation like the following (see Definition 2.1.5 for notation).

Example 1.5.15. *For a definable set $A \subset \mathbb{R}^n$ there exists a homeomorphism $\phi : A_a \times (0, a] \rightarrow A_{(0,a]}$. Moreover the homeomorphism can be chosen so that for $x \in A_a$, $y \in (0, a)$, the projection onto the second factor of $\phi(x, y)$ is y , and $\phi(A_a, a) = A_a$.*

1.5.4 Complexity Notation

In this thesis we use some notation that is not standard to Complexity theory. The “ O ” notation that follows is as normal, but the “ o ” notation is defined differently (taken from [2]), and Ω is a function defined later in the text, and is nothing to do with complexity.

We use the notation, whenever a more explicit bound is not more useful.

Definition 1.5.16. We say $f(x) \leq O(g(x))$ if there exists a positive real number M and an $x_0 \in \mathbb{R}$ such that

$$|f(x)| \leq M|g(x)|$$

for all $x > x_0$.

The following notation allows us to express how “small” certain expressions are, and is taken from [2]

Definition 1.5.17. For an expression $f = \sum_i (a_i \delta^{r_i})$, where $a_i \neq 0$ and δ is taken to be very small, we denote by $o(f)$ the smallest value of r_i .

1.5.5 Sard’s Theorem

The following is taken from [14]:

Definition 1.5.18. An n -cube $C \subset \mathbb{R}^n$ of edge $\lambda > 0$ is a product $C = I_1 \times \dots \times I_n$ of closed intervals of length λ . The measure of C is λ^n . A subset $X \subset \mathbb{R}^n$ has measure zero if for every $\varepsilon > 0$ it can be covered by a family of n -cubes, the sum of whose measures is less than ε .

Lemma 1.5.19. *A countable union of sets of measure zero has measure zero.*

Proof. Trivial. □

Theorem 1.5.20 (Sard's Theorem). *Let M, N be manifolds of dimensions m, n and $f : M \rightarrow N$ a C^r map. If $r > \max\{0, m - n\}$ then the set of critical values of f has measure zero in N .*

1.5.6 Basic concepts in Algebraic Topology

For a topological space X , we denote by $H_i(X)$ its i -th homology group, and by $b_i(X) = \text{rank } H_i(X)$ its i -th Betti number. We denote by $b(X) = \sum_i b_i(X)$ the sum of the Betti numbers of X .

Lemma 1.5.21. *The number of connected components of a non-empty closed and bounded definable set S is equal to $b_0(S)$*

Proof. See for example Proposition 6.26 of [2]. □

Since in this thesis we spend very little time dealing with homotopy groups, we usually use the symbol π for projections. However, in the few cases where we must mention homotopy groups, we follow the standard convention of using $\pi_i(X)$ to denote the i -th homotopy group of X , but clearly explain the situation to avoid any confusion.

The symbol \simeq denotes homotopy equivalence. If $Y \subset X$, then $\text{closure}(Y)$ denotes its closure in X , and \bar{Y} its complement. It is important to note that this departs from usual conventions, but is necessary due to the complex expressions that follow.

1.5.7 Significantly Small Real Numbers

We now give a definition for the \ll symbol.

Definition 1.5.22. Let $\mathcal{P} = \mathcal{P}(\varepsilon_0, \dots, \varepsilon_l)$ be a predicate (property) over $(0, 1)^{l+1}$. We say that property \mathcal{P} holds for

$$0 < \varepsilon_0 \ll \varepsilon_1 \ll \dots \ll \varepsilon_l \ll 1$$

if there exist definable functions $f_k : (0, 1)^{l-k} \rightarrow (0, 1)$, where $k = 0, \dots, l$ (with f_l being a positive constant), such that \mathcal{P} holds for any sequence $\varepsilon_0, \dots, \varepsilon_l$ satisfying

$$0 < \varepsilon_k < f_k(\varepsilon_{k+1}, \dots, \varepsilon_l) \text{ for } k = 0, \dots, l.$$

1.5.8 Mayer-Vietoris Inequalities

Let S_1 and S_2 be closed and bounded definable sets. Then there exists the following long exact sequence of homology groups:

$$\dots \rightarrow H_i(S_1 \cap S_2) \rightarrow H_i(S_1) \oplus H_i(S_2) \rightarrow H_i(S_1 \cup S_2) \rightarrow H_{i-1}(S_1 \cap S_2) \rightarrow \dots$$

see for example [13], [3].

From this, the following inequalities are easily derived:

$$b_i(S_1) + b_i(S_2) \leq b_i(S_1 \cup S_2) + b_i(S_1 \cap S_2) \quad (1.5.1)$$

$$b_i(S_1 \cap S_2) \leq b_i(S_1) + b_i(S_2) + b_{i+1}(S_1 \cup S_2), \quad (1.5.2)$$

and if $i > 0$

$$b_i(S_1 \cup S_2) \leq b_i(S_1) + b_i(S_2) + b_{i-1}(S_1 \cap S_2). \quad (1.5.3)$$

If $i = 0$ we have $b_0(S)$ is equal to the number of connected components of S , so we have

$$b_0(S_1 \cup S_2) \leq b_0(S_1) + b_0(S_2).$$

More generally we have:

Lemma 1.5.23 (Mayer-Vietoris inequality). *Let $X_1, \dots, X_m \subset [-1, 1]^n$ be all open or all closed in $[-1, 1]^n$. Then*

$$b_i \left(\bigcup_{1 \leq j \leq n} X_j \right) \leq \sum_{J \subset \{1, \dots, n\}} b_{i-|J|+1} \left(\bigcap_{j \in J} X_j \right)$$

and

$$b_i \left(\bigcap_{1 \leq j \leq n} X_j \right) \leq \sum_{J \subset \{1, \dots, n\}} b_{i+|J|-1} \left(\bigcup_{j \in J} X_j \right),$$

where b_i is the i -th Betti number.

Proof. See [12]. □

1.6 Complexity

In [4] the following axiomatic complexity metric was presented:

Definition 1.6.1. A *complexity* on $\mathbb{R}[X_1, \dots, X_n]$ is a map $c : \mathbb{R}[X_1, \dots, X_n] \rightarrow \mathbb{N}$ satisfying the following axioms:

- (c1) if P is constant, then $c(P) = 0$; $c(X_i) = 1$ ($1 \leq i \leq n$);
- (c2) $c(P + Q) \triangleleft (c(P), c(Q))$; $c(PQ) \triangleleft (c(P), c(Q)) \forall P, Q \in \mathbb{R}[X_1, \dots, X_n]$;
- (c3) $c(\partial P / \partial X_i) \triangleleft c(P) \forall P \in \mathbb{R}[X_1, \dots, X_n]$;
- (c4) for $P_1, \dots, P_n \in \mathbb{R}[X_1, \dots, X_n]$, let $n(S)$ be the number of non-degenerate solutions of the system $S : P_1 = \dots = P_n = 0$, then $n(S) \triangleleft (n, c(P_1), \dots, c(P_n))$.

The \triangleleft notation was defined as follows:

Definition 1.6.2. If $g : E \rightarrow \mathbb{N}$ and $f : E \rightarrow F$ are set maps, we write $g \triangleleft f$ if there is a set map $h : F \rightarrow \mathbb{N}$, “effectively” computable, such that $g(x) \leq h(f(x)) \forall x \in E$.

One of the main contributions of this thesis is that I present a new definition of an axiomatic complexity metric, that improves on the preceding in the following ways:

- We allow different classes of functions, rather than simply considering polynomials
- We allow multivariate complexity – the complexity of polynomials can simply be measured by degree, but other functions (for example Pfaffian) may have more than a single complexity measure
- Functions must be specified in place of the rather vague \triangleleft notation
- We specify that adding a constant to a function and multiplying a function by a non-zero constant cannot change complexity

The new axiomatic complexity metric is as follows:

Definition 1.6.3. Let $H \subset \mathbb{R}^n$ be a definable open set, and consider a set D of differentiable functions from H to \mathbb{R} in an o-minimal structure over the reals, such that D is closed under addition, multiplication and partial derivatives. Fix a positive integer m . A *complexity* on D is a pair (c, T) , where $c = (c_1, \dots, c_m)$ with $c_1, \dots, c_m : D \rightarrow \mathbb{N}$, and $T = (T_1, \dots, T_m)$ with $T_i = (t_{i,+}, t_{i,\times}, t_{i,\partial}, \{t_n\}_{1 \leq n < \infty})$, such that:

- (c0) $t_{i,+}, t_{i,\times} : \mathbb{N}^{2m} \rightarrow \mathbb{N}$, $t_{i,\partial} : \mathbb{N}^m \rightarrow \mathbb{N}$, $t_n : \mathbb{N}^{nm} \rightarrow \mathbb{N}$
- (c1) If $F \in D$ is constant, then $c_i(F) = 0$; $c_i(X_i) = 1$ ($1 \leq i \leq m$);
- (c2) For all $F, G \in D$,

$$c_i(F + G) \leq t_{i,+}(c_1(F), \dots, c_m(F), c_1(G), \dots, c_m(G)) \text{ for all } i = 1, \dots, m$$

and

$$c_i(FG) \leq t_{i,\times}(c_1(F), \dots, c_m(F), c_1(G), \dots, c_m(G)) \text{ for all } i = 1, \dots, m.$$

Furthermore if $F \in D$ is constant, then $c_i(F + G) = c_i(G)$, and if F is also non-zero then $c_i(FG) = c_i(G)$.

- (c3) For all $F \in D$, $c_i\left(\frac{\partial F}{\partial x_j}\right) \leq t_{i,\partial}(c_1(F), \dots, c_m(F))$;
- (c4) For $F_1, \dots, F_n \in D$, let χ be the number of solutions in \mathbb{R}^n of the system $F_1 = \dots = F_n = 0$. Then $\chi \leq t_n(c_1(F_1), \dots, c_m(F_1), \dots, c_1(F_n), \dots, c_m(F_n))$.

We will let t_+ denote $(t_{1,+}, \dots, t_{m,+})$, t_\times denote $(t_{1,\times}, \dots, t_{m,\times})$, and t_∂ denote $(t_{1,\partial}, \dots, t_{m,\partial})$. We will also sometimes use $c(F)$ to denote $(c_1(F), \dots, c_m(F))$.

Example 1.6.4 (The Degree of a Polynomial). Let $H = \mathbb{R}^n$, $m = 1$, $D = \mathbb{R}[X_1, \dots, X_n]$, with $P, P_i \in D$, we have $c(P) = (c_1(P)) = (\deg(P))$, and $t_{1,+} = \max(c_1(P_1), c_1(P_2))$, $t_{1,\times} = c_1(P_1) + c_1(P_2)$, $t_{1,\partial} = c_1(P) - 1$, and by Bezout's theorem ([17] Lemma 1), $t_n = c_1(P_1)c_1(P_2) \cdots c_1(P_n)$.

Example 1.6.5 (Pfaffian Functions). *Let $H \subset \mathbb{R}^n$ be an open domain (for example $(0, 1)^n$), D be the set of Pfaffian functions, and take $m = 3$. Let $f_1, \dots, f_n \in D$ have degrees $c(f_i) = (c_1, c_2, c_3) = (\alpha_i, \beta_i, r_i)$ for $1 \leq i \leq n$. Then from Lemma 1.5.7 we can take*

$$c(f_1 + f_2) = (\max\{\alpha_1, \alpha_2\}, \max\{\beta_1, \beta_2\}, r_1 + r_2), \text{ so}$$

$$t_{1,+} = \max\{\alpha_1, \alpha_2\}, t_{2,+} = \max\{\beta_1, \beta_2\}, t_{3,+} = r_1 + r_2;$$

$$c(f_1 f_2) = (\max\{\alpha_1, \alpha_2\}, \beta_1 + \beta_2, r_1 + r_2), \text{ so}$$

$$t_{1,\times} = \max\{\alpha_1, \alpha_2\}, t_{2,\times} = \beta_1 + \beta_2, t_{3,\times} = r_1 + r_2.$$

If f_1 and f_2 are defined by the same Pfaffian chain of order r we can further refine the above to

$$t_{3,+} = t_{3,\times} = r.$$

We also have from Lemma 1.5.8

$$c\left(\frac{\partial f_1}{\partial x_j}\right) = (\alpha_1, \alpha_1 + \beta_1 - 1, r_1), \text{ so}$$

$$t_{1,\partial} = \alpha_1, t_{2,\partial} = \alpha_1 + \beta_1 - 1, t_{3,\partial} = r_1.$$

If we take $\max_i\{\alpha_i\} = \alpha$ and $\max_i\{r_i\} = r$, Khovanskii's bound states the number of non-degenerate solutions of $f_1 = \dots = f_n = 0$ is bounded by

$$t_n = 2^{r(r-1)/2} \beta_1 \dots \beta_n (\min\{n, r\} \alpha + \beta_1 + \dots + \beta_n - n + 1)^r$$

(see Section 1.5.2).

To avoid overcomplicating future expressions, we also define the following:

Definition 1.6.6. For a complexity (c, T) , we define $t_{+,s} : \mathbb{N}^s \rightarrow \mathbb{N}$ and $t_{\times,s} : \mathbb{N}^s \rightarrow \mathbb{N}$ inductively:

$$t_{+,2} = t_+$$

$$t_{+,s}(c_1, \dots, c_s) = t_+(t_{+,s-1}(c_1, \dots, c_{s-1}), c_s).$$

For the definition of $t_{\times,s}$, replace $+$ by \times .

We define for repeated arguments to these functions:

$$t_{+,s}^*(m) = t_{+,s}(m, \dots, m)$$

and define $t_{\times,s}^*(m)$ similarly.

1.6.1 The Functions κ and γ

We will define some functions, whose backgrounds will be explained in full at a later stage.

The first arises when we compute the complexity of a function, after rotating the coordinates so that certain specifications are satisfied. Let $\kappa = (\kappa_1, \dots, \kappa_m)$, where for $1 \leq i \leq m$,

$$\kappa_i(c(F)) = t_{i+}(t_{i\partial}(c_1(F), \dots, c_m(F)), t_{i\times}(0, t_{i\partial}(c_1(F), \dots, c_m(F))))$$

The second function comes from when we are trying to find critical points on a hypersurface: we want the function F to be zero, along with certain combinations of partial derivatives, taking into account the rotation above. In the following, remember t_n has nm arguments:

$$\gamma(n, c(F)) = t_n(c_1(F), \dots, c_m(F), \kappa(c(F)), \kappa(c(F)), \dots, \kappa(c(F)))$$

Example 1.6.7 (The Degree of a Polynomial). *In the case of Example 1.6.4, if we have a polynomial P of degree d , then $\kappa(c(P)) = d - 1$, and $\gamma(n, c(P)) = d(d - 1)^{n-1}$.*

Example 1.6.8 (Pfaffian Functions). *Consider the setting of Example 1.6.5, where we deal with Pfaffian functions. Take such a function f with complexity (α, β, r) , then $\kappa(c(f)) = (\alpha, \alpha + \beta - 1, r)$, and*

$$\begin{aligned} \gamma(n, c(f)) &= t_n(c(f), \kappa(c(f)), \dots, \kappa(c(f))) \\ &= 2^{r(r-1)/2} \beta (\alpha + \beta - 1)^{n-1} (\min\{n, r\} \alpha + \beta + (n-1)(\alpha + \beta - 1) - n + 1)^r \\ &= 2^{r(r-1)/2} \beta (\alpha + \beta - 1)^{n-1} (\min\{n, r\} \alpha + n\beta + (n-1)\alpha - 2n + 2)^r \end{aligned}$$

Chapter 2

Sets defined by equalities and non-strict inequalities

2.1 Sets defined by equalities

2.1.1 Introduction

We follow the methods of Milnor to produce a bound for the sum of the Betti numbers of a set defined by equalities. We construct a hypersurface that has a “nice” projection map, and show that when moving a plane orthogonally to the axis of this projection map, Betti numbers can only increase or decrease at critical points of this projection map. The number of critical points of this projection map can be found using normal analytic methods, and thus a relationship between algebra and topology has been found. We now move to find the link between this hypersurface and the area it bounds.

In the following, let (c, T) be a complexity on a set D .

2.1.2 Main Result

Our main result in this section is:

Theorem 2.1.1. *Let f_1, \dots, f_m be functions in D , and let $S = \{f_1 = \dots = f_m = 0\}$. Then the sum of the Betti numbers $b(S)$ of S satisfies*

$$b(S) \leq \frac{\gamma(n, c(f_1^2 + \dots + f_m^2 + |x|^2))}{2}.$$

Example 2.1.2 (The Degree of a Polynomial). *Using the complexity described in Example 1.6.4, if the functions f_i have maximum degree d , then $f_1^2 + \dots + f_n^2 + |x|^2$ has maximum degree $2d$, and $\gamma = 2d(2d - 1)^{n-1}$, and $b(S) \leq d(2d - 1)^{n-1}$, which is the well-known Thom-Milnor bound.*

Example 2.1.3 (Pfaffian Functions). *Consider the setting of Example 1.6.5, and allow functions f_i to have a common Pfaffian chain of order r , and have complexity (α_i, β_i, r) . Then f_i^2 has complexity $(\alpha_i, 2\beta_i, r)$, and $f_1^2 + \dots + f_n^2 + |x|^2$ has complexity*

$$(\max_i \{\alpha_i\}, 2\max_i \{\beta_i\}, r) = (\alpha, 2\beta, r).$$

Therefore

$$\begin{aligned} b(S) &\leq \gamma(k, c(F))/2 \\ &= 2^{r(r-1)/2} \beta (\alpha + 2\beta - 1)^{n-1} (\min\{n, r\} \alpha + 2n\beta + (n-1)(\alpha) - 2n + 2)^r. \end{aligned}$$

This is equivalent to a bound given in [22].

Definition 2.1.4. Define $\pi : \mathbb{R}^n \rightarrow \mathbb{R}$ to be the projection onto the X_1 -axis, sending $x = (x_1, \dots, x_n) \in \mathbb{R}^n$ to $x_1 \in \mathbb{R}$.

We will also use the following notation, taken from [2]:

Definition 2.1.5. For $S \subset \mathbb{R}^n$, and $X \subset \mathbb{R}$, let S_X denote $S \cap \pi^{-1}(X)$. We also use the abbreviations S_x , $S_{<x}$, and $S_{\leq x}$ for $S_{\{x\}}$, $S_{(-\infty, x)}$, and $S_{(-\infty, x]}$ respectively.

2.1.3 Projections and Morse Functions

In this section we show that we can rotate the coordinates to ensure that π is a *Morse function*. We closely follow the working of [2].

Definition 2.1.6. Let $W = \{f = 0\}$ be a compact, non-singular (i.e. $\text{Grad}(f(x))|_W \neq 0$ at every point $x \in W$) hypersurface. Let $p \in W$ be a critical point of π . We can choose (X_2, \dots, X_n) to be a local system of coordinates in a sufficiently small neighbourhood of p . More precisely, we have an open neighbourhood $U \subset \mathbb{R}^{n-1}$ of $p' = (p_2, \dots, p_n)$ and a map $\phi : U \rightarrow \mathbb{R}$ such that, with $x' = (x_2, \dots, x_n)$ and

$$\Phi(x') = (\phi(x'), x') \in W,$$

Φ is a diffeomorphism from U to $\Phi(U)$.

Definition 2.1.7. The critical point p is non-degenerate if the $(n-1) \times (n-1)$ Hessian matrix

$$\left[\frac{\partial^2 \phi}{\partial X_i \partial X_j}(p') \right], 2 \leq i, j \leq n$$

is invertible.

Definition 2.1.8. The function π is called a Morse function if all its critical points are non-degenerate, and all critical values are pair-wise distinct.

Before moving onto the next Lemma, we define the following basic concepts:

Definition 2.1.9 (Orthogonal matrix). An matrix is *orthogonal* if its transpose is equal to its inverse. When considered as a linear map, an orthogonal matrix corresponds to an isometry (distance preserving map) between spaces, for example a rotation or reflection.

Definition 2.1.10 (Changes of coordinates). An *orthogonal change of coordinates* consists of multiplying the existing coordinate system by an orthogonal matrix. This allows manipulation of the “viewpoint” of the space without affecting topological invariants. A particular example of this is rotation about a point – in this case the orthogonal matrix must also have determinant 1.

Lemma 2.1.11. *Let f be a function in D , and W be as defined in Definition 2.1.6. Up to an orthogonal change of coordinates, the projection π onto the X_1 -axis is a Morse function on W .*

To prove this, we first need the following.

Definition 2.1.12 (Gauss map). Define the *Gauss map* from the hypersurface $W \subset \mathbb{R}^n$ to the unit sphere $S^{n-1} \subset \mathbb{R}^n$ by:

$$g(x) = \frac{\text{Grad}(f(x))}{|\text{Grad}(f(x))|}.$$

Lemma 2.1.13. *Let $p \in W$ be a critical point of π . Then p is not a critical point of the Gauss map if and only if p is a non-degenerate critical point of π .*

Proof. Since p is a critical point of π , $g(p) = (\pm 1, 0, \dots, 0)$. Using Definition 2.1.6, for $x' \in U$, $x = \Phi(x') = (\phi(x'), x')$, and applying the chain rule,

$$\frac{\partial f}{\partial X_i}(x) + \frac{\partial f}{\partial X_1}(x) \frac{\partial \phi}{\partial X_i}(x') = 0, 2 \leq i \leq n.$$

Thus

$$g(x) = \pm \frac{1}{\sqrt{1 + \sum_{i=2}^n \left(\frac{\partial \phi}{\partial X_i}(x') \right)^2}} \left(-1, \frac{\partial \phi}{\partial X_2}(x'), \dots, \frac{\partial \phi}{\partial X_n}(x') \right).$$

Taking the partial derivative with respect to X_i of the j -th coordinate g_j of g , $2 \leq i, j \leq n$, and evaluating at p we obtain

$$\frac{\partial g_j}{\partial X_i}(p) = \pm \frac{\partial^2 \phi}{\partial X_j \partial X_i}(p'), 2 \leq i, j \leq n.$$

The matrix $M = \left[\frac{\partial g_j}{\partial X_i}(p) \right]$, $2 \leq i, j \leq n$ is invertible if and only if p is a non-degenerate critical point of ϕ by Definition 2.1.7. If M is invertible then it is non-zero, and g is non-critical at p . \square

Lemma 2.1.14. *There exists an orthogonal change of coordinates such that π has only non-degenerate critical points.*

Proof. Consider the Gauss map g from W to the unit sphere; we want $(\pm 1, 0, \dots, 0)$ to be regular values of g . If this is not the case, we find a rotation of coordinates as follows. Sard’s

theorem tells us that the set of critical values of g on the unit sphere has measure zero. Take U to be the upper hemisphere of the unit sphere, (note that U does not have measure zero). Consider two subsets, one R containing regular values of g and one C containing critical values, such that $U = R \cup C$. Since a countable union of sets of measure zero also has measure zero, we know that R does not have measure zero. Consider the set directly opposite R on the lower hemisphere. This set also does not have measure zero, so contains at least one regular value of g . Take this regular value, and the opposite point on the upper hemisphere, which is also a regular value, and rotate the coordinates so that the points on W that were taken by g to these values are now taken to $(\pm 1, 0, \dots, 0)$.

We can now find a rotation of coordinates to ensure $(\pm 1, 0, \dots, 0)$ are not critical values of g . The claim now follows from Lemma 2.1.13. \square

Definition 2.1.15. For a compact, non-singular hypersurface W in \mathbb{R}^n , we define χ to be the number of critical points of π on W .

We now only need to prove that, changing the coordinates if necessary, critical values are pair-wise distinct. The first step in doing this uses our definition of complexity to find the number of critical points.

Lemma 2.1.16. *Under the conditions of Lemma 2.1.11, suppose that π has only non-degenerate critical points. Then χ is finite, and bounded by $\gamma(n, c(f))$.*

Proof. Lemma 2.1.14 tells us that we may need to rotate the coordinate system to ensure critical points are non-degenerate. If all critical points are non-degenerate without any rotation, the definition of a critical point means that at each point $\frac{\partial f}{\partial X_2} = \dots = \frac{\partial f}{\partial X_n} = 0$.

However, as stated, the above workings may have required an orthogonal change of coordinates (i.e. multiplication of the coordinate system by an orthogonal matrix). This means that each partial derivative of the new coordinate system can be expressed as a linear combination of partial derivatives of the old system. As the matrix is orthogonal, it is invertible, and we can solve this system of $n - 1$ equations to express each of $\frac{\partial f}{\partial X_i}$ for $i = 2 \dots n$ as $\frac{\partial f}{\partial X_1}$ multiplied by a real number. In other words we have

$$\begin{aligned} \frac{\partial f}{\partial X_2} &= \lambda_2 \frac{\partial f}{\partial X_1} \\ &\vdots \\ \frac{\partial f}{\partial X_n} &= \lambda_n \frac{\partial f}{\partial X_1}, \end{aligned}$$

for $\lambda_2 \dots \lambda_n \in \mathbb{R}$.

In addition to these $n - 1$ equations, we must also require $f = 0$ (as the critical point is on the hypersurface). We now have a system of n equations, and each solution corresponds to a critical point of π . As the critical points are non-degenerate, every solution is non-singular, and we can use the definition of complexity to bound the number of solutions.

We have

$$\chi \leq t_n \left(c(f), c \left(\frac{\partial f}{\partial X_2} - \lambda_2 \frac{\partial f}{\partial X_1} \right), \dots, c \left(\frac{\partial f}{\partial X_n} - \lambda_n \frac{\partial f}{\partial X_1} \right) \right).$$

Also, remembering that all constants have complexity 0, we have

$$c_j \left(\frac{\partial f}{\partial X_i} - \lambda_i \frac{\partial f}{\partial X_1} \right) \leq t_{j+}(t_{j\partial}(c(F)), t_{j\times}(0, t_{j\partial}(c(F)))) ,$$

and hence we have

$$\chi \leq \gamma(n, c(f))$$

□

Suppose that π has only non-degenerate critical points. Lemma 2.1.16 tells us that these are finite in number. We can suppose that without loss of generality all the critical points have distinct X_2 coordinates, making if necessary an orthogonal change of coordinates in the variables X_2, \dots, X_n only.

Lemma 2.1.17. *Let $\delta > 0$ be sufficiently small. The set S of points $\bar{p} = (\bar{p}_1, \dots, \bar{p}_n) \in W = \{f = 0\}$ with gradient vector $\text{Grad}(f)(\bar{p})$ proportional to $(1, \delta, 0, \dots, 0)$ is finite. Its number of elements is equal to χ , the number of critical points of π . Moreover, there is a point \bar{p} of S infinitesimally close to every critical point p of π , and p and \bar{p} are either both non-degenerate, or both degenerate.*

Proof. Note that, modulo the orthogonal change of variable

$$X'_1 = X_1 + \delta X_2, X'_2 = X_2 - \delta X_1, X'_i = X_i \text{ for } i \geq 3,$$

a point \bar{p} such that $\text{Grad}(f)(\bar{p})$ is proportional to $(1, \delta, 0, \dots, 0)$ is a critical point of the projection π' onto the X'_1 axis, and the corresponding critical value of π' is $\bar{p}_1 + \delta \bar{p}_2$.

Since $W = \{f = 0\}$ is bounded, a point $\bar{p} \in W$ always has an image by $\lim_{\delta \rightarrow 0}$. If \bar{p} is such that $\text{Grad}(f)(\bar{p})$ is proportional to $(1, \delta, 0, \dots, 0)$, then $\text{Grad}(f)(\lim_{\delta \rightarrow 0}(\bar{p}))$ is proportional to $(1, 0, 0, \dots, 0)$, and thus $p = \lim_{\delta \rightarrow 0}(\bar{p})$ is a critical point of π . Suppose without loss of generality that $\text{Grad}(f)(p) = (1, 0, 0, \dots, 0)$. Since p is a non-degenerate critical point of π , Lemma 2.1.13 implies that there is a neighbourhood U of $p' = (p_2, \dots, p_n)$, such that $g \circ \Phi$ is a diffeomorphism from U to a neighbourhood of $(1, 0, 0, \dots, 0) \in S^{n-1}(0, 1)$. Denoting by g' the inverse of the restriction of g to $\Phi(U)$, and considering $g' : g(\Phi(U)) \rightarrow \Phi(U)$, there is a unique $\bar{p} \in \Phi(U)$ such that $\text{Grad}(f)(\bar{p})$ is proportional to $(1, \delta, 0, \dots, 0)$. Moreover, denoting by J the Jacobian of g' , $J(1, 0, 0, \dots, 0) = t$ is a non-zero real number. Thus the signature of the Hessian (see Definition 2.1.7) at p and \bar{p} coincide, and they are either both non-degenerate, or both degenerate. □

Proof of Lemma 2.1.11. Since J is the Jacobian of g' , $J(1, 0, 0, \dots, 0) = t$ is a non-zero real number, $\lim_{\delta \rightarrow 0}(J(y)) = t$ for every $y \in S^{n-1}(0, 1)$ infinitesimally close to $(1, 0, \dots, 0)$. Using the mean value theorem,

$$o(|\bar{p} - p|) = o \left(\left| \frac{1}{\sqrt{1 + \delta^2}} (1, \delta, 0, \dots, 0) - (1, 0, 0, \dots, 0) \right| \right) = 1.$$

Thus $o(\bar{p}_i - p_i) \geq 1$, $i \geq 1$.

Let $b_{i,j} = \frac{\partial^2 \phi}{\partial X_i \partial X_j}(p)$, $2 \leq i \leq n$, $2 \leq j \leq n$. Taylor's formula for p at ϕ gives

$$\bar{p}_1 = p_1 + \sum_{2 \leq i \leq n, 2 \leq j \leq n} b_{i,j}(\bar{p}_i - p_i)(\bar{p}_j - p_j) + c,$$

with $o(c) \geq 2$. Thus $o(\bar{p}_1 - p_1) \geq 2$.

It follows that the critical value of π' at \bar{p} is $\bar{p}_1 + \delta \bar{p}_2 = p_1 + \delta p_2 + w$, with $o(w) > 1$.

Let us take two critical values $v = p_1 + \delta p_2 + w$, $v' = p'_1 + \delta p'_2 + w'$. If $p_1 \neq p'_1$, then we can assume that δ is chosen to be small enough for $v \neq v'$. If $p_1 = p'_1$, we know that all values of p_2 are distinct, and w is negligible, so we also have $v \neq v'$.

Thus all critical values of π' on W are distinct. □

2.1.4 A Bounding Hypersurface

Before proving Theorem 2.1.1, we need a few preliminary results.

Firstly we follow the method used in [2] to bound the sum of the Betti numbers of a compact non-singular hypersurface in terms of the number of critical points of the projection.

Lemma 2.1.18. *If $W = \{f = 0\}$ is a compact, non-singular hypersurface, and π is a Morse function, then the sum of the Betti numbers of W is bounded by χ .*

Proof. Let v_1, \dots, v_l be the critical values of π , and $p_1 < p_2 < \dots < p_l$ the corresponding critical points such that $\pi(p_i) = v_i$. We will show that $b(W_{\leq v_i}) \leq i$.

We move along the X_1 -axis. First note that $W_{\leq v_1} = \{p_1\}$, and hence $b(W_{\leq v_1}) = 1$. Morse Theory (for details see [14], [2]) tells us that between critical points of π , homotopy type doesn't change, i.e. $W_{\leq v_i + \varepsilon}$ is homotopy equivalent to $W_{\leq v_{i+1} - \varepsilon}$ for any small enough $\varepsilon > 0$. We thus only need to consider what happens when we pass through critical points. Morse Theory also tells us that the homotopy type of $W_{\leq v_i + \varepsilon}$ is that of the union of $W_{\leq v_i - \varepsilon}$ with a topological ball.

From the Mayer-Vietoris Inequalities (1.5.3) we have that if S_1 and S_2 are two closed sets with non-empty intersection, and if $i > 0$ then

$$b_i(S_1 \cup S_2) \leq b_i(S_1) + b_i(S_2) + b_{i-1}(S_1 \cap S_2).$$

For a closed and bounded set S , $b_0(S)$ is equal to the number of connected components of S . Since $S_1 \cap S_2 \neq \emptyset$, for $i = 0$ we have

$$b_0(S_1 \cup S_2) \leq b_0(S_1) + b_0(S_2) - 1.$$

We also have, for $\lambda > 1$

$$b_0(\overline{B_\lambda}) = b_0(S^{\lambda-1}) = b_{\lambda-1}(S^{\lambda-1}) = 1,$$

and for $0 < i < \lambda - 1$ we have

$$b_i(S^{\lambda-1}) = 0.$$

Therefore, for $\lambda > 1$, attaching a λ -ball can increase b_λ by at most one, and none of the other Betti numbers can increase.

For $\lambda = 1$, $b_{\lambda-1}(S^{\lambda-1}) = b_0(S^0) = 2$. In this case, b_1 can increase by at most one – adding an edge to a graph can increase the number of cycles by at most one, and therefore adding $\overline{B_1}$ to a set can increase the number of “holes”, i.e. b_1 by 1. For $i > 1$, $b_{i-1}(S \cap \overline{B_1}) = 0$, and $b_i(\overline{B_1}) = 0$, so $b_i(S \cup \overline{B_1}) \leq b_i(S)$, and none of the other Betti numbers can increase.

Thus $b(W_{\leq v_{i+1}}) \leq b(W_{\leq v_i}) + 1$, and the result follows. \square

We again closely follow the methods used in [2], this time to bound the sum of the Betti numbers of the set bounded by the above hypersurface.

Lemma 2.1.19. *If $W = \{f = 0\}$ is again as above, $K = \{f \geq 0\}$ is bounded by W , and π is a Morse function, then the sum of the Betti numbers of K is bounded by $\chi/2$.*

Proof. As above, let v_1, \dots, v_l be the critical values of π , and $p_1 < p_2 < \dots < p_l$ the corresponding critical points such that $\pi(p_i) = v_i$. Let J be the subset of $\{1, \dots, l\}$ such that the direction of $\text{Grad}(f)(p_i)$ belongs to K . We will show that $b(K_{\leq v_i})$ is less than or equal to the number of $j \in J$ with $j \leq i$.

First note that $K_{\leq v_1} = \{p_1\}$, and hence $b(K_{\leq v_1}) = 1$. Morse Theory (as above) tells us that $K_{\leq v_{i+1}-\varepsilon}$ is homotopic to $K_{v_i+\varepsilon}$ for any small enough $\varepsilon > 0$, and thus $b(K_{\leq v_{i+1}-\varepsilon}) = b(K_{v_i+\varepsilon})$.

Morse Theory also tells us that $K_{\leq v_i+\varepsilon}$ has the same homotopy type as $K_{\leq v_i-\varepsilon}$ if $i \notin J$, and has the homology type of the union of $K_{\leq v_i-\varepsilon}$ with a topological ball if $i \in J$ (see [2], Proposition 7.20). It follows that

$$b(K_{\leq v_i+\varepsilon}) = \begin{cases} b(K_{\leq v_i-\varepsilon}) & \text{if } i \notin J \\ b(K_{\leq v_i-\varepsilon}) + 1 & \text{if } i \in J. \end{cases}$$

By switching the direction of the X_1 -axis if needed, we can always ensure that the number of points in J is at most $\chi/2$. \square

2.1.5 Proof of main result

We can summarise three preceding results as follows:

Corollary 2.1.20. *From Lemmas 2.1.16, 2.1.18, and 2.1.19 we now have the following:*

$$\begin{aligned} b(W) &\leq \chi \\ b(K) &\leq \chi/2 \\ \chi &\leq \gamma(n, c(f)) \end{aligned}$$

and can deduce

$$\begin{aligned} b(W) &\leq \gamma(n, c(f)) \\ b(K) &\leq \gamma(n, c(f))/2. \end{aligned}$$

We can now prove the main result of this section, again largely following the methods used in [2].

Proof of Theorem 2.1.1. Since S is a definable set, by Theorem 1.5.4 we can choose a large r such that $r^2 - |x|^2$ is always strictly positive for $x \in S$, and the sum of the Betti numbers of $S \cap B(0, r)$ coincides with that of S .

Let $G(x) = \frac{f_1^2 + \dots + f_m^2}{r^2 - |x|^2}$. By Sard's theorem, the set of critical values of G is finite. Hence there is a positive $a \in \mathbb{R}$ so that no $b \in (0, a)$ is a critical value of G , and thus the set $W_b = \{x \in \mathbb{R}^n : f_1^2 + \dots + f_m^2 + b(|x|^2 - r^2) = 0\}$ is a non-singular hypersurface in \mathbb{R}^n . To see this, setting $F(x, b) = f_1^2 + \dots + f_m^2 + b(|x|^2 - r^2)$, we have that if for some x ,

$$F(x, b) = \frac{\partial F}{\partial X_1}(x, b) = \dots = \frac{\partial F}{\partial X_n}(x, b) = 0$$

then

$$G(x) = b \text{ and } \frac{\partial G}{\partial X_1}(x) = \dots = \frac{\partial G}{\partial X_n}(x) = 0,$$

which means that b is a critical value of G , a contradiction.

Moreover, W_b is the boundary of the closed and bounded set $K_b = \{x \in \mathbb{R}^n : F(x, b) \leq 0\}$. By Lemma 2.1.18 the sum of the Betti numbers of W_b is less than or equal to $\gamma(n, c(F))$, and using Lemma 2.1.19 the sum of the Betti numbers of K_b is at most half that of W_b .

We now claim that $S \cap B(0, r)$ is homotopy equivalent to K_b for all small enough $b > 0$. We replace b in the definition by a new variable t , and consider the set $K \subset \mathbb{R}^{n+1}$ defined by $\{(x, t) \in \mathbb{R}^{n+1} : F(x, t) \leq 0\}$. Let π_X (respectively π_T) denote the projection map onto the x (respectively t) coordinates.

Clearly, $S \cap B(0, r) \subset K_b$. By Hardt's triviality (Theorem 1.5.14), for all small enough $b > 0$, there exists a homeomorphism $\phi : K_b \times (0, b] \rightarrow K \cap \pi_T^{-1}((0, b])$, such that $\pi_T(\phi(x, s)) = s$ and $\phi(S \cap B(0, r), s) = S \cap B(0, r)$ for all $s \in (0, b]$.

Let $H : K_b \times [0, b] \rightarrow K_b$ be the map defined by $H(x, s) = \pi_X(\phi(x, s))$ for $s > 0$, and $H(x, 0) = \lim_{s \rightarrow 0+} \pi_X(\phi(x, s))$. Let $h : K_b \rightarrow S \cap B(0, r)$ be the map $H(x, 0)$, and $i : S \cap B(0, r) \rightarrow K_b$ be the inclusion map. Using the homotopy H , we see that $i \circ g \sim Id_{K_b}$, and $g \circ i \sim Id_{S \cap B(0, r)}$, which shows that $S \cap B(0, r)$ is homotopy equivalent to K_b as claimed.

Hence,

$$b(S \cap B(0, r)) = b(K_b) \leq \frac{1}{2}b(W_b) \leq \frac{\gamma(n, c(F))}{2}.$$

Since b and r are constants, we have $c(f_1^2 + \dots + f_m^2 + b(|x|^2 - r^2)) = c(f_1^2 + \dots + f_m^2 + |x|^2)$, and produce the desired result. □

2.2 Sets defined by multiple non-strict inequalities

We next move to the case where our set is defined by multiple equations and non-strict inequalities. The next Lemma follows the method shown in [17] to construct a set defined in terms of a collection of non-strict inequalities, and uses the method of proof of homotopy equivalence shown in Proposition 7.28 of [2].

Lemma 2.2.1. *Let f_1, f_2, \dots, f_p be functions in D , and let $S = \{f_1 \geq 0, f_2 \geq 0, \dots, f_p \geq 0\}$. Then the sum of the Betti numbers of S $b(S)$ satisfies*

$$b(S) \leq \frac{\gamma(n, c(|x|^2 f_1 f_2 \cdots f_p))}{2}.$$

Proof. The set S is homotopy equivalent to $S \cap B(0, r)$ for sufficiently large r , so we can add to the system the inequality $f_0 = r^2 - |x|^2 \geq 0$ to allow us to assume that the set is compact. For $\varepsilon \geq \delta > 0$, consider the set $L_{\varepsilon, \delta}$ defined by

$$f_0 + \varepsilon \geq 0, \dots, f_p + \varepsilon \geq 0 \tag{2.2.1}$$

and

$$(f_0 + \varepsilon)(f_1 + \varepsilon) \cdots (f_p + \varepsilon) - \delta \geq 0. \tag{2.2.2}$$

This set is compact, and we obtain the boundary by setting the left hand side of (2.2.2) equal to zero. Given ε we can choose δ so that the bounding surface is smooth, and therefore we can use Lemma 2.1.18 to bound the sum of the Betti numbers of this bounding surface, and use Lemma 2.1.19 to bound the sum of the Betti numbers of $L_{\varepsilon, \delta}$.

Setting $M_{\varepsilon, \delta} = (f_0 + \varepsilon)(f_1 + \varepsilon) \cdots (f_p + \varepsilon) - \delta$, we have

$$b(L_{\varepsilon, \delta}) \leq \frac{\gamma(n, c(M_{\varepsilon, \delta}))}{2}.$$

We can use the definition of the complexity c to find the complexity $c(M_{\varepsilon, \delta})$. From (c1) we know the complexity of an expression does not depend on the value of constants, so $c(M_{\varepsilon, \delta})$ does not depend on the values of ε and δ and we can rename all instances of this $c(M)$.

It is clear that $S \cap B(0, r) \subset L_{\varepsilon, \delta}$. Set

$$L = \{(x, \varepsilon, \delta) \in \mathbb{R}^{n+2} : f_0(x) + \varepsilon \geq 0, \dots, f_p(x) \geq 0, (f_0(x) + \varepsilon)(f_1(x) + \varepsilon) \cdots (f_p(x) + \varepsilon) - \delta \geq 0\}.$$

Let π_X, π_E, π_Δ be the projection onto the x, ε and δ coordinates respectively. By Hardt's triviality (Theorem 1.5.14) there exists a homeomorphism

$$\phi : L_{\varepsilon, \delta} \times (0, \varepsilon] \times (0, \delta] \rightarrow L \cap \pi_E^{-1}((0, \varepsilon]) \cap \pi_\Delta^{-1}((0, \delta])$$

such that $\pi_E(\phi(x, a, b)) = a$, $\pi_\Delta(\phi(x, a, b)) = b$, and

$$\phi(S \cap B(0, r), a, b) = S \cap B(0, r)$$

for all $a \in (0, \varepsilon]$, $b \in (0, \delta]$.

Let

$$G : L_{\varepsilon, \delta} \times [0, \varepsilon] \times [0, \delta] \rightarrow L_{\varepsilon, \delta}$$

be the map defined by

$$G(x, a, b) = \begin{cases} \pi_X(\phi(x, a, b)) & \text{if } a, b > 0, \\ \lim_{a \rightarrow 0^+} \pi_X(\phi(x, a, b)) & \text{if } a = 0, b > 0, \\ \lim_{b \rightarrow 0^+} \pi_X(\phi(x, a, b)) & \text{if } a > 0, b = 0, \\ \lim_{a, b \rightarrow 0^+} \pi_X(\phi(x, a, b)) & \text{if } a, b = 0. \end{cases}$$

Let $g : L_{\varepsilon, \delta} \rightarrow S \cap B(0, r)$ be the map $G(x, 0, 0)$, and $i : S \cap B(0, r) \rightarrow L_{\varepsilon, \delta}$ be the inclusion map. We can see that $i \circ g \sim Id_{L_{\varepsilon, \delta}}$, and $g \circ i \sim Id_{S \cap B(0, r)}$, so $S \cap B(0, r)$ is homotopy equivalent to $L_{\varepsilon, \delta}$ as desired.

Hence:

$$b(S) = b(S \cap B(0, r)) = b(L_{\varepsilon, \delta}) \leq \frac{\gamma(n, c(M))}{2}.$$

Finally we can simplify the resulting expression by removing constants. □

Example 2.2.2 (The Degree of a Polynomial). *Using the complexity described in Example 1.6.4, if the functions f_i have maximum degree d , then the function $|x|^2 f_1 \cdots f_p$ has maximum degree $2 + pd$, and $\gamma = (2 + pd)(1 + pd)^{n-1}$, and $b(S) \leq (2 + pd)(1 + pd)^{n-1}/2$, which is the same result shown in Theorem 3 of [17].*

Example 2.2.3 (Pfaffian Functions). *Consider the setting of Example 1.6.5, and allow functions f_i to have degree (α_i, β_i, r) . Set $\max_i \{\alpha_i\} = \alpha$, $\max_i \{\beta_i\} = \beta$. Then*

$$c(|x|^2 f_1 \cdots f_p) = (\alpha, 2 + \sum_{1 \leq i \leq p} \beta_i, r),$$

and

$$\begin{aligned} b(S) &\leq \gamma/2 \\ &\leq 2^{r(r-1)/2-1} (2 + p\beta)(\alpha + 2 + p\beta - 1)^{n-1} (\min\{n, r\}\alpha + n(2 + p\beta) + (n-1)\alpha - 2n + 2)^r \\ &= 2^{r(r-1)/2-1} (2 + p\beta)(\alpha + 1 + p\beta)^{n-1} (\min\{n, r\}\alpha + np\beta + (n-1)\alpha + 2)^r. \end{aligned}$$

This is equivalent to results given in [22].

It is easy to see that any collection of equations can be replaced by two non-strict inequalities, by setting $F = f_1^2 + \dots + f_m^2 = 0$ and considering $F \geq 0$ and $-F \geq 0$. We can now combine this observation with the above result to get a theorem considering the union of any collection of equations and non-strict inequalities.

Theorem 2.2.4. *Let $S = \{f_1 = 0, \dots, f_m = 0, f_{m+1} \geq 0, \dots, f_p \geq 0\}$, where $f_i \in D$. Then we have*

$$b(S) \leq \frac{1}{2} \gamma \left(n, c(|x|^2 (f_1^2 + \dots + f_m^2)^2 f_{m+1} \dots f_p) \right).$$

Proof. Follows from above discussion. □

Chapter 3

Sets defined by a Boolean combination of functions

3.1 Preliminaries

The following notation is taken from [3]. Let G and F be finite sets of definable functions from $H \rightarrow \mathbb{R}$, where $H \subset \mathbb{R}^n$ is a definable open set. A sign condition on F is an element of $\{0, 1, -1\}^F$. Let F have s elements.

We define the sets Z and Z_r by

$$Z = \left\{ x \in \mathbb{R}^n : \bigwedge_{g \in G} g(x) = 0 \right\},$$

$$Z_r = Z \cap B(0, r).$$

Let the dimension of Z be n' .

The realisation of the sign condition σ over Z , $Real(\sigma, Z)$ is the definable set

$$\left\{ x \in \mathbb{R}^n : \bigwedge_{g \in G} g(x) = 0 \wedge \bigwedge_{f \in F} \text{sign}(f(x)) = \sigma(f) \right\}.$$

For each $0 \leq i \leq n'$, we denote the sum of the i -th Betti numbers of the realisations of all sign conditions of F on Z by $b_i(F, G)$.

3.1.1 The Function Ω

Definition 3.1.1. Let

$$\begin{aligned} \omega(F, G) &= \max_i (b(\text{Real}(f_i = \delta, Z_r))) \\ &= \max_i \left(\frac{\gamma(n, c(f_i^2 + \sum_{g \in G} g^2 + |x|^2))}{2} \right); \end{aligned}$$

we then define $\Omega(F, G)$ to be

$$\max(\omega(F, G), b(Z_r)) = \max\left(\omega(F, G), \frac{\gamma(n, c(\sum_{g \in G} g^2 + |x|^2))}{2}\right).$$

Example 3.1.2 (The Degree of a Polynomial). *Again in the situation of Example 1.6.4, with the degree of the functions in F and G being bounded by d , we have*

$$\begin{aligned}\Omega(F, G) &= \frac{\gamma(n, 2d)}{2} \\ &= d(2d - 1)^{n-1}\end{aligned}$$

Example 3.1.3 (Pfaffian Functions). *Let the functions in F and G have a common Pfaffian chain of order r . For $h \in F, G$, let $\max\{c_1(h)\} = \alpha$, $\max\{c_2(h)\} = \beta$. Then*

$$\begin{aligned}\Omega(F, G) &= \frac{\gamma(n, (\alpha, 2\beta, r))}{2} \\ &= 2^{r(r-1)/2} \beta (\alpha + 2\beta - 1)^{n-1} (\min\{n, r\} \alpha + 2n\beta + (n-1)\alpha - 2n + 2)^r\end{aligned}$$

3.1.2 Main Result

Our main result in this chapter is this:

Theorem 3.1.4. *Let $Y \subset \mathbb{R}^n$ be a set defined by a Boolean combination of definable functions $\{f_1, \dots, f_s\}$. Then*

$$b(Y) \leq \sum_{i=0}^n \sum_{j=1}^{n-i} \binom{2s^2 + 1}{j} 6^j \Omega(F', \emptyset)$$

where F' is a set of functions containing all the elements f_i and $|x|^2$.

We initially consider the realisation of sign conditions of sets of functions, and use this to show how to bound the Betti numbers of closed sets. We then use an inductive construction to find a closed set from any set defined by a Boolean combination of functions, and prove that in fact this closed set we have found is homotopy equivalent to our initial set. We can then use our bound for closed sets to produce the above bound for arbitrary Boolean combinations. The following sections owe much to [3], [1] and [2].

3.2 Realisation of Sign Conditions

Our main result in this section is the following:

Theorem 3.2.1.

$$b_i(F, G) \leq \sum_{j=0}^{n'-i} \binom{s}{j} 4^j \Omega(F, G).$$

To prove this result, we define a set W_0 to be the union of some sets we can easily bound, and bound the Betti numbers of W_0 using the Mayer-Vietoris Inequalities. We define a set W_1 that has W_0 as its boundary, and use the Mayer-Vietoris inequalities and the previous result to bound the Betti numbers of W_1 . Finally we relate these sets back to $b_i(F, G)$, and prove the main result. These methods are the same as those used in [3], but with the generalisation to functions defined in o-minimal structures (instead of simply polynomials), which results in replacing the expression $d(2d - 1)^{n-1}$ with $\Omega(F, G)$.

3.2.1 Sign Conditions

The following lemmas also follow the method of proof shown in [3], this time adapting for our more general circumstances.

Set $W_0 = \bigcup_{1 \leq i \leq j} (\text{Real}(f_i^2(f_i^2 - \delta^2) = 0, Z_r))$ for sufficiently small δ .

Lemma 3.2.2. $b_i(W_0) \leq (4^j - 1)\Omega(F, G)$

Proof. Each of the sets $\text{Real}(f_i^2(f_i^2 - \delta^2) = 0, Z_r)$ is the disjoint union of three sets defined by definable equations, namely $\text{Real}(f_i = 0, Z_r)$, $\text{Real}(f_i = \delta, Z_r)$, and $\text{Real}(f_i = -\delta, Z_r)$. Moreover, from the Mayer-Vietoris Inequalities (Lemma 1.5.23), each of the Betti numbers of their union is bounded by the sum of the Betti numbers of all possible non-empty sets that can be obtained by taking, for $1 \leq l \leq j$, l -ary intersections of these sets. The number of possible l -ary intersections is $\binom{j}{l}$. Each such intersection is a disjoint union of 3^l sets. The sum of the Betti numbers of each of these sets is bounded by $\Omega(F, G)$ by Definition 3.1.1, thus

$$b_i(W_0) \leq \sum_{l=1}^j \binom{j}{l} 3^l \Omega(F, G) = (4^j - 1)\Omega(F, G)$$

□

Now set $W_1 = \bigcup_{1 \leq i \leq j} (\text{Real}(f_i^2(f_i^2 - \delta^2) \geq 0, Z_r))$ for sufficiently small δ .

Lemma 3.2.3. $b_i(W_1) \leq (4^j - 1)\Omega(F, G) + b_i(Z_r)$

Proof. Let $Q_i = f_i^2(f_i^2 - \delta^2)$, and

$$K = \text{Real} \left(\bigwedge_{1 \leq i \leq j} (Q_i \leq 0) \vee \bigvee_{1 \leq i \leq j} (Q_i = 0), Z_r \right).$$

Now applying inequality (1.5.1), noting that $W_1 \cup K = Z_r$ and $W_1 \cap K = W_0$. We get that $b_i(W_1) \leq b_i(W_1 \cap K) + b_i(W_1 \cup K) = b_i(W_0) + b_i(Z_r)$. We conclude using Lemma 3.2.2. □

Now let $S_i = \text{Real}(f_i^2(f_i^2 - \delta^2) \geq 0, Z_r)$, and S be the intersection of S_i . Then

Lemma 3.2.4. $b_i(F, G) = b_i(S)$.

Proof. Consider a sign condition σ on F such that, without loss of generality,

$$\begin{aligned}\sigma(f_i) &= 0 \text{ if } i = 1, \dots, j, \\ \sigma(f_i) &= 1 \text{ if } i = j+1, \dots, l, \\ \sigma(f_i) &= -1 \text{ if } i = l+1, \dots, s\end{aligned}$$

and denote by $\overline{Real i}(\sigma)$ the subset of Z_r defined by

$$\bigwedge_{i=1, \dots, j} f_i(x) = 0 \wedge \bigwedge_{i=j+1, \dots, l} f_i(x) \geq \delta \wedge \bigwedge_{i=l+1, \dots, s} f_i(x) \leq -\delta$$

It follows from Hardt's triviality (Theorem 1.5.14) that $b_i(\overline{Real i}(\sigma)) = b_i(\sigma)$, with δ sufficiently small. Note that S is the disjoint union of the $\overline{Real i}(\sigma)$ (for the σ realisable sign conditions) so that $\sum_{\sigma} b_i(\sigma) = b_i(S)$. On the other hand, $\sum_{\sigma} b_i(\sigma) = b_i(F, G)$. \square

We now can prove our most substantial result

proof of Theorem 3.2.1. Using the Mayer-Vietoris Inequalities (Lemma 1.5.23), Lemma 3.2.3, and Lemma 2.1.18 which implies, for all $i < n'$, $b_i(Z_r) + b_{n'}(Z_r) \leq b(Z_r) \leq \Omega(F, G)$, we find that

$$\begin{aligned}b_i(S) &\leq b_{n'}(S^{\emptyset}) + \sum_{j=1}^{n'-i} \sum_{J \subset \{1, \dots, s\}, \#(J)=j} (4^j - 1) \Omega(F, G) + b_i(Z_r) + b_{n'}(Z_r) \\ &\leq b_{n'}(S^{\emptyset}) + \sum_{j=1}^{n'-i} \sum_{J \subset \{1, \dots, s\}, \#(J)=j} 4^j \Omega(F, G) \\ &\leq \sum_{j=0}^{n'-i} \binom{s}{j} 4^j \Omega(F, G)\end{aligned}$$

\square

3.3 Closed Sets

In this section we prove a bound for the Betti numbers of closed (and by Alexander's duality open) sets. We first define closed formulae (which are used to describe closed sets), and proceed to define sets based on these sets, slightly modified by small real numbers. We show how one can bound the Betti numbers of the conjunction of two formulae by the sum of the Betti numbers of the conjunction of these with a third formula that ranges over a given set, and then use this to show how to bound the Betti numbers of a formula in terms of the sum of the Betti numbers of some sets that are in some sense a subset of the original set. We next define sets W_0 and W_1 similar to those in the above section, and proceed to bound the Betti numbers of these sets. These are then shown to be exactly related to the bounds we have found for closed formulae above, and a bound for any closed set easily follows. Again much of the following is taken from [3], with the same modification as above, again resulting in the introduction of $\Omega(F, G)$. We

show (in an example) how this bound reduces to that given in [3], in the polynomial, degree case, and show how a parallel to $s^{n'} O(d)^n$ can be found.

An (F, G) -closed formula is defined as follows:

- For each $f \in F$, $\bigvee_{g \in G} g = 0 \wedge f = 0$, $\bigvee_{g \in G} g = 0 \wedge f \geq 0$, $\bigvee_{g \in G} g = 0 \wedge f \leq 0$ are (F, G) -closed formulae.
- If ϕ_1 and ϕ_2 are (F, G) -closed formulae, then $\phi_1 \wedge \phi_2$ and $\phi_1 \vee \phi_2$ are (F, G) -closed formulae.

We denote by $b(\phi)$ the maximum sum of the Betti numbers of any realisation of this formula.

Let $F = \{f_1, \dots, f_s\}$ be α -minimal functions, and let $\delta_1, \dots, \delta_s$ be sufficiently small real numbers, with $\delta_{i+1} < \delta_i$. Define the following:

$$\begin{aligned} F_{>i} &= \{f_{i+1}, \dots, f_s\} \\ \Sigma_i &= \{f_i = 0, f_i = \delta_i, f_i = -\delta_i, f_i \geq 2\delta_i, f_i \leq -2\delta_i\} \\ \Sigma_{\leq i} &= \{\psi : \psi = \bigwedge_{j=1, \dots, i} \psi_j, \psi_j \in \Sigma_j\} \end{aligned}$$

Lemma 3.3.1. *For every (F, G) -closed formula ϕ , and every $\psi \in \Sigma_{\leq i}$,*

$$b(\phi \wedge \psi) \leq \sum_{\pi \in \Sigma_{i+1}} b(\phi \wedge \psi \wedge \pi).$$

Proof. Consider

$$\begin{aligned} \phi_1 &= \phi \wedge \psi \wedge (f_{i+1}^2 - \delta_{i+1}^2 \geq 0) \\ \phi_2 &= \phi \wedge \psi \wedge (0 \leq f_{i+1}^2 \leq \delta_{i+1}^2). \end{aligned}$$

Clearly a realisation of $\phi \wedge \psi$ is also a realisation of $\phi_1 \vee \phi_2$. From inequality (1.5.3), we then have

$$b(\phi \wedge \psi) \leq b(\phi_1) + b(\phi_2) + b(\phi_1 \wedge \phi_2).$$

Now

$$\begin{aligned} b(\phi_1 \wedge \phi_2) &= b(\phi \wedge \psi \wedge (f_{i+1}^2 - \delta_{i+1}^2 \geq 0) \wedge (0 \leq f_{i+1}^2 \leq \delta_{i+1}^2)) \\ &= b(\phi \wedge \psi \wedge (f_{i+1} = \delta_{i+1})) + b(\phi \wedge \psi \wedge (f_{i+1} = -\delta_{i+1})). \end{aligned}$$

By Hardt's triviality (Theorem 1.5.14), if we set $M_t = \{x \in \text{Real}(\phi \wedge \psi) : f_{i+1} = t\}$, then there exists $t_0 \in \mathbb{R}$ such that $M_{[-t_0, 0) \cup (0, t_0]} = \{x \in \text{Real}(\phi \wedge \psi) : f_{i+1} \in [-t_0, 0) \cup (0, t_0]\}$ and $([-t_0, 0) \times M_{-t_0}) \cup ((0, t_0] \times M_{t_0})$ are homeomorphic, and moreover the homeomorphism can be chosen to be the identity on the fibers M_{t_0} and M_{-t_0} . This clearly implies that $M_{[\delta, t_0]} = \{x \in \text{Real}(\phi \wedge \psi) : t_0 \geq f_{i+1} \geq \delta\}$ and $M_{[2\delta, t_0]} = \{x \in \text{Real}(\phi \wedge \psi) : t_0 \geq f_{i+1} \geq 2\delta\}$ are homeomorphic. Hence,

$$b(\phi_1) = b(\phi \wedge \psi \wedge (f_{i+1} \geq 2\delta_{i+1})) + b(\phi \wedge \psi \wedge (f_{i+1} \leq -2\delta_{i+1})).$$

Note that $M_0 = \text{Real}(\phi \wedge \psi \wedge (f_{i+1} = 0))$, and $M_{[-\delta, \delta]} = \text{Real}(\phi_2)$. By Hardt's Triviality (Theorem 1.5.14), for every $0 \leq u \leq 1$ there is a fiber-preserving homeomorphism ϕ_u from $M_{[-\delta, -u\delta]}$ to $[-\delta, -u\delta] \times M_{-u\delta}$ (resp. a homeomorphism ψ_u from $M_{[u\delta, \delta]}$ to $[u\delta, \delta] \times M_{u\delta}$). We define a continuous homotopy m from $M_{[-\delta, \delta]}$ to M_0 as follows:

- $m(0, -)$ is $\lim_{\delta_{i+1}}$,
- for $0 < u \leq 1$, $m(u, -)$ is the identity on $M_{[-u\delta, u\delta]}$ and sends $M_{[-\delta, -u\delta]}$ (resp. $M_{[u\delta, \delta]}$) to $M_{-u\delta}$ (resp. $M_{u\delta}$) by ϕ_u (resp. ψ_u) followed by the projection on $M_{u\delta}$ (resp. $M_{-u\delta}$).

Thus $b(M_{[-\delta, \delta]}) = b(M_0)$, and $b(\phi \wedge \psi) \leq \sum_{\pi \in \Sigma_{i+1}} b(\phi \wedge \psi \wedge \pi)$. \square

Lemma 3.3.2. *For every (F, G) -closed formula ϕ ,*

$$b(\phi) \leq \sum_{\psi \in \Sigma_{\leq s}, \text{Real}(\psi) \subset \text{Real}(\phi)} b(\psi).$$

Proof. Starting from the formula ϕ , apply Lemma 3.3.1 with ψ the empty formula. Now, repeatedly apply Lemma 3.3.1 to the terms appearing on the right-hand side of the inequality obtained, noting that for any $\psi \in \Sigma_{\leq s}$,

- either $\text{Real}(\phi \wedge \psi) = \text{Real}(\psi)$ and $\text{Real}(\psi) \subset \text{Real}(\psi)$,
- or $\text{Real}(\phi \wedge \psi) = \emptyset$.

\square

We now use these results to prove a bound on the Betti numbers of closed sets.

Let $F = \{f_1, \dots, f_j\}$ be o-minimal functions, and let

$$q_i = f_i^2(f_i^2 - \delta_i^2)^2(f_i^2 - 4\delta_i^2).$$

Let W_0 be the union of the set $\text{Real}(q_i = 0, Z_r)$ and W_1 be the union of $\text{Real}(q_i \geq 0, Z_r)$, with $1 \leq i \leq j$.

Note that $W_1 = \bigcup_{\psi \in \Sigma_{\leq s}} \text{Real}(\psi)$.

Lemma 3.3.3. $b_i(W_0) \leq (6^j - 1)\Omega(F, G)$

Proof. The set $\text{Real}((f_i^2(f_i^2 - \delta_i^2)^2(f_i^2 - 4\delta_i^2)), Z_r)$ is the disjoint union of

$$\begin{aligned} &\text{Real}(f_i = 0, Z_r), \\ &\text{Real}(f_i = \delta_i, Z_r), \\ &\text{Real}(f_i = -\delta_i, Z_r), \\ &\text{Real}(f_i = 2\delta_i, Z_r), \text{ and} \\ &\text{Real}(f_i = -2\delta_i, Z_r). \end{aligned}$$

Moreover, the i -th Betti numbers of their union W_0 is bounded by the sum of the Betti numbers of all possible non-empty sets that can be obtained by taking intersections of these sets using the Mayer-Vietoris Inequalities (Lemma 1.5.23).

The number of possible l -ary intersections is $\binom{j}{l}$. Each such intersection is a disjoint union of 5^l algebraic sets. The i -th Betti number of each of these algebraic sets is bounded by $\Omega(F, G)$ by Definition 3.1.1. Thus,

$$b_i(W_0) \leq \sum_{l=1}^j 5^l \Omega(F, G) = (6^j - 1) \Omega(F, G)$$

□

Lemma 3.3.4. $b_i(W_1) \leq (6^j - 1) \Omega(F, G) + b_i(Z_r)$.

Proof. Let $S = \text{Real}(\bigwedge_{1 \leq i \leq j} q_i \leq 0 \vee \bigvee_{1 \leq i \leq j} q_i = 0, Z_r)$. Now, $W_1 \cup S = Z_r$, and $W_1 \cap S = W_0$. Using inequality (1.5.1) we deduce that $b_i(W_1) \leq b_i(W_1 \cap S) + b_i(W_1 \cup S) = b_i(W_0) + b_i(Z_r)$. We conclude using Lemma 3.3.3. □

Lemma 3.3.5. $\sum_{\psi \in \Sigma_{<s}} b(\psi) \leq \sum_{j=0}^{n'-1} \binom{s}{j} 6^j \Omega(F, G)$

Proof. Since for all $i < n'$, $b_i(Z_r) + b_{n'}(Z_r) \leq \Omega(F, G)$ by Definition 3.1.1, we have that

$$\sum_{\psi \in \Sigma_{\leq s}} b(\psi) = b(W_1) \leq b_{n'}(Z_r) + \sum_{j=1}^{n'-1} \binom{s}{j} 6^j \Omega(F, G)$$

using the Mayer-Vietoris Inequalities (Lemma 1.5.23) and Lemma 3.3.4. Thus,

$$\sum_{\psi \in \Sigma_{\leq s}} b(\psi) \leq \sum_{j=0}^{n'-1} \binom{s}{j} 6^j \Omega(F, G)$$

□

And our main result:

Theorem 3.3.6. *Let G and F be finite sets of definable functions from $H \rightarrow \mathbb{R}$, where $H \subset \mathbb{R}^n$ is a definable open set. Let $|F| = s$, and $\dim\{\bigcup_{g \in G} g = 0\} = n' \leq n$. The sum of the Betti numbers of a closed set defined by non-strict inequalities on functions $f \in F$, and equations on functions $g \in G$, is bounded by*

$$\sum_{i=0}^{n'} \sum_{j=0}^{n'-i} \binom{s}{j} 6^j \Omega(F, G)$$

Proof. Follows from Lemmas 3.3.2 and 3.3.5. □

Example 3.3.7 (The Degree of a Polynomial). *In the polynomial, degree case we have $\Omega(F, G) = d(2d - 1)^{n-1}$, and we obtain Theorem 4.1 of [3], and the bound $s^{n'} O(d)^n$.*

Example 3.3.8 (Pfaffian Functions). *We have*

$$\Omega(F, G) = 2^{r(r-1)/2} \beta(\alpha + 2\beta - 1)^{n-1} (\min\{n, r\}\alpha + 2n\beta + (n-1)\alpha - 2n + 2)^r,$$

and from this we get the bound

$$s^{n'} 2^{r(r-1)/2} O(n\beta + \min\{n, r\}\alpha)^{n+r},$$

which is equivalent to Theorem 3.4 of [9].

From this we can deduce the following:

Corollary 3.3.9. *Let X be a set defined by a monotone Boolean combination (i.e. exclusively connectives \vee, \wedge are used, no negations) of definable functions containing only non-strict inequalities, then*

$$b(X) \leq O(s^n \gamma(k, c(\sum_i f_i^2 + |x|^2))).$$

Proof. Taking $G = \emptyset$ in Theorem 3.3.6 gives $n' = n$, and

$$\Omega(F, G) = \frac{\gamma(n, c(\sum_i f_i^2 + |x|^2))}{2}. \quad (3.3.1)$$

□

Corollary 3.3.10. *The above bound also applies to open sets defined by a monotone Boolean combination of definable functions containing only strict inequalities.*

Proof. Consider such a set, and in the Boolean combination replace each $<$ with \geq , $>$ with \leq , \cup with \cap , and \cap with \cup . The original Boolean combination is equivalent to the negation of this new formula (which we can bound using the previous Corollary). But Alexander's Duality tells us the set defined by this new Boolean combination must have the same Betti sum as the original. □

3.4 Arbitrary Boolean Combinations

We now move to bound sets defined by an arbitrary Boolean combination of equations and inequalities, both strict and non-strict. Our method of proof is taken from the second edition of [2]. For a given set X defined by such a combination, we inductively define a series of new sets X^0, \dots, X^{t+1} , and eventually show that $X \simeq X^{t+1}$. The latter is a closed set, which can be dealt with using the method of the previous section. This homotopy equivalence is shown via considering various sets defined in the inductive process, and showing that certain inclusion relations are in fact homotopy equivalences. We can use this result to directly calculate an explicit bound for a set defined by any Boolean combination, and we provide an example in the polynomial, degree case which reduces exactly to the bound given in our source. We also show how to find a bound parallel to $O(s^2 d)^n$.

Definition 3.4.1. Let F be a finite set of definable functions with t elements, and let S be a bounded F -closed set.

Set $\text{SIGN}(S)$ to be the set of realisable sign conditions of F whose realisation is contained in S .

For $\sigma \in \text{SIGN}(F)$, we call the number of elements in $\{f \in F : \sigma(f) = 0\}$ the level of σ .

Let $\varepsilon_1, \varepsilon_2, \dots, \varepsilon_{2t-1}, \varepsilon_{2t}$ be sufficiently small real numbers.

We now start constructing a set that will approximate our original set. We first define the elements that will make up this construction.

Note in the following that the c in σ_+^c stands for closed, and the o in σ_+^o stands for open.

Definition 3.4.2. For each level m , $0 \leq m \leq t$, we set $\text{SIGN}_m(S)$ to be the subset of $\text{SIGN}(S)$ of elements of level m .

Given $\sigma \in \text{SIGN}_m(F, S)$, we set

$$\begin{aligned} \text{Reali}(\sigma_+^c) = S \cap \{ & \{-\varepsilon_{2m} \leq f \leq \varepsilon_{2m} \text{ for each } f \text{ such that } \sigma f = 0\} \cup \\ & \cup \{f \geq 0 \text{ for each } f \text{ such that } \sigma f = 1\} \cup \\ & \cup \{f \leq 0 \text{ for each } f \text{ such that } \sigma f = -1\} \} \end{aligned}$$

and

$$\begin{aligned} \text{Reali}(\sigma_+^o) = S \cap \{ & \{-\varepsilon_{2m-1} < f < \varepsilon_{2m-1} \text{ for each } f \text{ such that } \sigma f = 0\} \cup \\ & \cup \{f > 0 \text{ for each } f \text{ such that } \sigma f = 1\} \cup \\ & \cup \{f < 0 \text{ for each } f \text{ such that } \sigma f = -1\} \} \end{aligned}$$

Note that

$$\begin{aligned} \text{Reali}(\sigma) & \subset \text{Reali}(\sigma_+^c), \\ \text{Reali}(\sigma) & \subset \text{Reali}(\sigma_+^o). \end{aligned}$$

We now define the construction of our approximating set.

Definition 3.4.3. Let $X \subset S$ be a set defined on F such that

$$X = \bigcup_{\sigma \in \Sigma} \text{Reali}(\sigma)$$

with $\Sigma \subset \text{SIGN}(S)$. Set $\Sigma_m = \Sigma \cap \text{SIGN}_m(S)$.

We define a sequence of sets $X^m \subset \mathbb{R}^n$ for $0 \leq m \leq t$ inductively by

- $X^0 = X$

- For $0 \leq m \leq t$,

$$X^{m+1} = \left(X^m \cup \bigcup_{\sigma \in \Sigma_m} \text{Reali}(\sigma_+^c) \right) \setminus \bigcup_{\sigma \in \text{SIGN}_m(S) \setminus \Sigma_m} \text{Reali}(\sigma_+^o)$$

We set $X' = X^{t+1}$.

Theorem 3.4.4. $X \simeq X'$

Before proving this, we need to introduce the following new sets, which we define inductively.

Definition 3.4.5. For each p with $0 \leq p \leq t+1$, we define sets $Y_p, Z_p \subset \mathbb{R}^n$ as follows

- We set

$$Y_p^p = X \cup \bigcup_{\sigma \in \Sigma_p} \text{Reali}(\sigma_+^c)$$

and

$$Z_p^p = Y_p^p \setminus \bigcup_{\sigma \in \text{SIGN}_p(S) \setminus \Sigma_p} \text{Reali}(\sigma_+^o)$$

- For $p \leq m \leq t+1$ we define

$$Y_p^{m+1} = \left(Y_p^m \cup \bigcup_{\sigma \in \Sigma_m} \text{Reali}(\sigma_+^c) \right) \setminus \bigcup_{\sigma \in \text{SIGN}_m(S) \setminus \Sigma_m} \text{Reali}(\sigma_+^o)$$

and

$$Z_p^{m+1} = \left(Z_p^m \cup \bigcup_{\sigma \in \Sigma_m} \text{Reali}(\sigma_+^c) \right) \setminus \bigcup_{\sigma \in \text{SIGN}_m(S) \setminus \Sigma_m} \text{Reali}(\sigma_+^o)$$

- Define $Y_p = Y_p^{t+1} \subset \mathbb{R}^n$, and $Z_p = Z_p^{t+1} \subset \mathbb{R}^n$.

Note that

- $X = Y_{t+1} = Z_{t+1}$, and
- $Z_0 = X'$.

Also note that for each p with $0 \leq p \leq t$,

- $Z_{p+1}^{p+1} \subset Y_p^p$, and
- $Z_p^p \subset Y_p^p$.

The following Lemma follows directly from the above definitions of Y_p and Z_p .

Lemma 3.4.6. For each p with $0 \leq p \leq t$,

- $Z_{p+1} \subset Y_p$, and
- $Z_p \subset Y_p$.

To prove Theorem 3.4.4, it suffices to prove that the above inclusions are actually homotopy equivalences.

Lemma 3.4.7. *For $1 \leq p \leq t$, $Y_p \simeq Z_{p+1}$.*

Proof. Let $Y_p(u) \subset \mathbb{R}^n$ be the set obtained by replacing ε_{2p} in the definition of Y_p by u , and for $u_0 > 0$ define

$$Y_p((0, u_0]) = \{(x, u) : x \in Y_p(u), u \in (0, u_0]\} \subset \mathbb{R}^{n+1}.$$

By Hardt's Triviality (Theorem 1.5.14), there exists $u_0 > 0$ and a homeomorphism

$$\psi : Y_p(u_0) \times (0, u_0] \rightarrow Y_p((0, u_0])$$

such that

1. $\pi_{k+1}(\psi(x, u)) = u$,
2. $\psi(x, u_0) = (x, u_0)$ for $x \in Y_p(u_0)$,
3. for all $u \in (0, u_0]$ and for every sign condition σ on

$$\bigcup_{f \in F} \{f, f \pm \varepsilon_{2t}, \dots, f \pm \varepsilon_{2p+1}\}$$

$\psi(\cdot, u)$ defines a homeomorphism of $\text{Real}(\sigma, Y_p(u_0))$ to $\text{Real}(\sigma, Y_p(u))$.

Now specify u_0 to be ε_{2p} , and set ϕ to be the map corresponding to ψ . For $\sigma \in \Sigma_p$, define:

$$\begin{aligned} \text{Real}(\sigma_{++}^o) &= \{-2\varepsilon_{2p} < f < \varepsilon_{2p} \text{ for all } f \text{ such that } \sigma(f) = 0\} \cup \\ &\cup \{f > -2\varepsilon_{2p} \text{ for all } f \text{ such that } \sigma(f) = 1\} \cup \\ &\cup \{f < 2\varepsilon_{2p} \text{ for all } f \text{ such that } \sigma(f) = -1\}. \end{aligned}$$

Let $\lambda : Y_p \rightarrow \mathbb{R}$ be a continuous definable function (for example piecewise linear would suffice) such that

- $\lambda(x) = 1$ on $Y_p \cap \bigcup_{\sigma \in \Sigma_p} \text{Real}(\sigma_+^c)$,
- $\lambda(x) = 0$ on $Y_p \setminus \bigcup_{\sigma \in \Sigma_p} \text{Real}(\sigma_{++}^o)$,
- $0 < \lambda(x) < 1$ otherwise.

We now consider a homotopy $Y_p \times [0, \varepsilon_{2p}] \rightarrow Y_p$ by defining

- $h(x, t) = \pi_{1\dots k} \circ \phi(x, \lambda(x)t + (1 - \lambda(x))\varepsilon_{2p})$ for $0 < t \leq \varepsilon_{2p}$
- $h(x, 0) = \lim_{t \rightarrow 0+} h(x, t)$ otherwise.

Note that this limit exists since S is closed and bounded. We now show that $h(x, 0) \in Z_{p+1}$ for all $x \in Y_p$. Let $x \in Y_p$ and $y = h(x, 0)$. We have two cases:

1. $\lambda(x) < 1$. Then we have $x \in Z_{p+1}$, and by property (3) of ϕ and the fact that $\lambda(x) < 1$, $y \in Z_{p+1}$.
2. $\lambda(x) \geq 1$. Let σ_y be the sign condition of F at y , and suppose that $y \notin Z_{p+1}$. We have two cases:
 - (a) $\sigma_y \in \Sigma$. Therefore $y \in X$, and there exists a $\tau \in \text{SIGN}_m(S) \setminus \Sigma_m$ with $m > p$ such that $y \in \text{Reali}(\tau_+^o)$.
 - (b) $\sigma_y \notin \Sigma$. In this case, taking $\tau = \sigma_y$, we have $\text{level}(\tau) > p$ and $y \in \text{Reali}(\tau_+^o)$. It follows from the definition of y and property (3) of ϕ that for all $m > p$ and for all $\rho \in \text{SIGN}_m(S)$:
 - $y \in \text{Reali}(\rho_+^o)$ implies that $x \in \text{Reali}(\rho_+^o)$
 - $x \in \text{Reali}(\rho_+^c)$ implies that $y \in \text{Reali}(\rho_+^c)$

Thus $x \notin Y_p$, a contradiction.

It follows that:

- $h(\cdot, \varepsilon_{2p}) : Y_p \rightarrow Y_p$ is the identity,
- $h(Y_p, 0) = Z_{p+1}$, and
- $h(\cdot, t)$ restricted to Z_{p+1} gives a homotopy between

$$h(\cdot, \varepsilon_{2p})|_{Z_{p+1}} = \text{id}_{Z_{p+1}}$$

and

$$h(\cdot, 0)|_{Z_{p+1}}.$$

Thus $Y_p \simeq Z_{p+1}$. □

Lemma 3.4.8. *For all p with $0 \leq p \leq t$, $Z_p \simeq Y_p$.*

Proof. For this proof define the following new sets for $u \in \mathbb{R}$:

- $Z'_p(u) \subset \mathbb{R}^n$ is the set obtained by replacing in the definition of Z_p , ε_{2j} by $\varepsilon_{2j} - u$, and ε_{2j-1} by $\varepsilon_{2j-1} + u$ for all $j > p$, and ε_{2p} by $\varepsilon_{2p} - u$ and ε_{2p-1} by u .

For all $u_0 > 0$, define

$$Z'_p((0, u_0]) = \{(x, u) : x \in Z'_p(u), u \in (0, u_0]\}.$$

- Let $Y'_p(u) \subset \mathbb{R}^n$ be the set obtained by replacing in the definition of Y_p , ε_{2j} by $\varepsilon_{2j} - u$ and ε_{2j-1} by $\varepsilon_{2j-1} + u$ for all $j > p$, and ε_{2p} by $\varepsilon_{2p} - u$.
- For $\sigma \in \text{SIGN}_m(S)$ with $m \geq p$, let $\text{Reali}(\sigma_+^c)(u) \subset \mathbb{R}^n$ denote the set obtained by replacing ε_{2m} by $\varepsilon_{2m} - u$ in the definition of $\text{Reali}(\sigma_+^c)$.

- For $\sigma \in \text{SIGN}_m(S)$ with $m > p$, let $\text{Reali}(\sigma_+^o)(u) \subset \mathbb{R}^n$ denote the set obtained by replacing ε_{2m-1} by $\varepsilon_{2m-1} + u$ in the definition of $\text{Reali}(\sigma_+^o)$.
- Finally, for $\sigma \in \text{SIGN}_p(S)$, let $\text{Reali}(\sigma_+^o)(u) \subset \mathbb{R}^n$ denote the set obtained by replacing ε_{2p-1} by u in the definition of $\text{Reali}(\sigma_+^o)$.

Note that by the definitions, for all $u, v \in \mathbb{R}$ with $0 < u \leq v$, $Z'_p(u) \subset Y'_p(u)$, $Z'_p(v) \subset Z'_p(u)$, $Y'_p(v) \subset Y'_p(u)$, and

$$\bigcup_{0 < s \leq u} Y'_p(s) = \bigcup_{0 < s \leq u} Z'_p(s).$$

Let $Z'_p = Z'_p(\varepsilon_{2p-1})$ and $Y'_p = Y'_p(\varepsilon_{2p-1})$. It is easy to see that $Y'_p \simeq Y_p$, and $Z'_p \simeq Z_p$. We will now prove that $Y'_p \simeq Z'_p$, which suffices to prove the Lemma.

Let $\mu : Y'_p \rightarrow \mathbb{R}$ be a definable map with

$$\mu(x) = \sup_{u \in (0, \varepsilon_{2p-1}]} \{u : x \in Z'_p(u)\}.$$

We will prove below (Lemma 3.4.9) that μ is continuous. Note that the definition of $Z'_p(u)$ (as well as $Y'_p(u)$) is more complicated than the natural one consisting of replacing ε_{2p-1} in the definition of Z_p by u , due to the fact that with the latter definition μ is not necessarily continuous.

We now construct a definable map

$$h : Y'_p \times [0, \varepsilon_{2p-1}] \rightarrow Y'_p$$

as follows. By Hardt's Triviality (Theorem 1.5.14) there exists $u_0 > 0$ and a homeomorphism

$$\psi : Z'_p(u_0) \times (0, u_0] \rightarrow Z'_p((0, u_0])$$

such that:

1. $\pi_{k+1}(\psi(x, u)) = u$,
2. $\psi(x, u_0) = (x, u_0)$ for $x \in Z'_p(u_0)$,
3. for all $u \in (0, u_0]$ and for all sign conditions σ of

$$\bigcup_{f \in F} \{f, f \pm \varepsilon_{2t}, \dots, f \pm \varepsilon_{2p+1}\},$$

the map $\psi(\cdot, u)$ restricts to a homeomorphism of $\text{Reali}(\sigma, Z'_p(u_0))$ to $\text{Reali}(\sigma, Z'_p(u))$.

Now specify that $u_0 = \varepsilon_{2p-1}$, and denote by ϕ the corresponding map

$$\phi : Z'_p \times (0, \varepsilon_{2p-1}] \rightarrow Z'_p((0, \varepsilon_{2p-1})).$$

Note that for all u with $0 < u \leq \varepsilon_{2p-1}$, ϕ gives a homeomorphism

$$\phi_u : Z'_p(u) \rightarrow Z'_p.$$

Hence for every pair u, u' with $0 < u \leq u' \leq \varepsilon_{2p-1}$, there exists a homeomorphism

$$\theta_{u,u'} : Z'_p(u) \rightarrow Z'_p(u')$$

obtained by composing ϕ_u with $\phi_{u'}^{-1}$. For $0 \leq u' \leq u \leq \varepsilon_{2p-1}$, let $\theta_{u,u'}$ be the identity map. It is clear that $\theta_{u,u'}$ moves continuously with u and u' .

For $x \in Y'_p$ and $t \in [0, \varepsilon_{2p-1}]$, now define

$$h(x, t) = \theta_{\mu(x), t}(x).$$

It is easy to verify from the definition of h and the properties of ϕ above that h is continuous and:

- $h(\cdot, 0) : Y'_p \rightarrow Y'_p$ is the identity map
- $h(Y'_p, \varepsilon_{2p-1}) = Z'_p$,
- $h(\cdot, t)$ restricts to a homeomorphism $Z'_p \times t \rightarrow Z'_p$ for every $t \in [0, \varepsilon_{2p-1}]$.

Thus, $Z_p \simeq Y_p$ □

Lemma 3.4.9. *For sufficiently small ε_{2p-1} , the definable map $\mu : Y'_p \rightarrow \mathbb{R}$ defined by*

$$\mu(x) = \sup_{u \in (0, \varepsilon_{2p-1}]} \{u : x \in Z'_p(u)\}$$

is continuous.

Proof. In order to prove the continuity of μ we show that for every point $x' \in Y'_p$, for all $\varepsilon_\mu > 0$ there exists $\delta_\mu > 0$ such that for all $x \in Y'_p$ with $|x - x'| < \delta_\mu$ we have $|\mu(x) - \mu(x')| < \varepsilon_\mu$.

Choose ε_μ . Since each function $f \in F$ is continuous, setting $\varepsilon := \varepsilon_\mu/2$ in the standard definition of continuity, for each f there exists $\delta_f > 0$ such that if $|x - x'| < \delta_f$ then $|f(x) - f(x')| < \varepsilon_\mu/2$. Set $\delta_\mu = \min_{f \in F} \{\delta_f\}$, and fix x and x' with $|x - x'| < \delta_\mu$. Let $u \in (0, \varepsilon_{2p-1}]$ be such that $x \in Z'_p(u)$. We show below that this implies that $x' \in Z'_p(u')$ for some u' satisfying $|u - u'| < \varepsilon_\mu$.

Let m be the largest integer such that there exists $\sigma \in \Sigma_m$ with $x \in \text{Reali}(\sigma_+^c)(u)$: since $x \in Z'_p(u)$ such an m must exist. We have two cases:

1. $m > p$: Let $\sigma \in \Sigma_m$ with $x \in \text{Reali}(\sigma_+^c)(u)$. Then by maximality of m , for all $f \in F$ with $\sigma(f) \neq 0$ we have $f(x) \neq 0$. Therefore $x' \in \text{Reali}(\sigma_+^c)(u')$ for all $u' < u - \max_{f \in F, \sigma(f)=0} |f(x) - f(x')| \leq u - \varepsilon_\mu/2$. We can thus choose u' such that $x' \in \text{Reali}(\sigma_+^c)(u')$ and $|u - u'| < \varepsilon_\mu$.

2. $m \leq p$: If $x' \notin Z'_p(u)$ then since $x' \in Y'_p \subset Y'_p(u)$,

$$x' \in \bigcup_{\sigma \in \text{SIGN}_p(F, S) \setminus \Sigma_p} \text{Reali}(\sigma_+^o)(u).$$

Let $\sigma \in \text{SIGN}_p(S) \setminus \Sigma_p$ be such that $x' \in \text{Reali}(\sigma_+^o)(u)$. We prove by contradiction that

$$\max_{f \in F, \sigma(f)=0} |f(x')| = u.$$

Assume the contrary. Since $x \notin \text{Reali}(\sigma_+^o)(u)$ by assumption, and $|x - x'| < \varepsilon_\mu$, there must exist $f \in F$ with $\sigma(f) \neq 0$ and $f(x) = 0$. Letting τ denote the sign condition defined by $\tau(f) = 0$ if $f(x) = 0$ and $\tau(f) = \sigma(f)$ otherwise, we have that $\text{level}(\tau) > p$ and x belongs to both $\text{Reali}(\tau_+^o)(u)$ and $\text{Reali}(\tau_+^c)(u)$. Now there are two cases:

- (a) $\tau \in \Sigma$: then the fact that $x \in \text{Reali}(\tau_+^c)(u)$ contradicts the choice of m , since $m \leq p$ and $\text{level}(\tau) > p$.
- (b) $\tau \notin \Sigma$: then x gets removed at the level of τ in the construction of $Z'_p(u)$, and hence $x \in \text{Reali}(\rho_+^c)(u)$ for some $\rho \in \Sigma$ with $\text{level}(\rho) > \text{level}(\tau) > p$. This again contradicts the choice of m .

Thus, $\max_{f \in F, \sigma(f)=0} |f(x')| = u$, and since

$$x' \notin \bigcup_{\sigma \in \text{SIGN}_p(C, S) \setminus \Sigma_p} \text{Reali}(\sigma_+^o)(u')$$

for all $u' < \max_{f \in F, \sigma(f)=0} |f(x')| = u$, we can choose u' such that $|u - u'| < \varepsilon_\mu$ and $x' \notin \bigcup_{\sigma \in \text{SIGN}_p(F, S) \setminus \Sigma_p} \text{Reali}(\sigma_+^o)(u')$.

In both cases we have $x' \in Z'_p(u')$ for some u' satisfying $|u - u'| < \varepsilon_\mu$. This means for $x \in Y'_p$ and u with $x \in Z'_p(u)$, and for x' with $|x' - x| < \delta_\mu$, there exists u' with $x' \in Z'_p(u')$, with $u' < u$ and $|u - u'| < \varepsilon_\mu$. We therefore have $\mu(x) - \mu(x') < \varepsilon_\mu$. Reversing the roles of x and x' in the above, we get $\mu(x') - \mu(x) < \varepsilon_\mu$. Hence $|\mu(x) - \mu(x')| < \varepsilon_\mu$, and μ is continuous. \square

Proof of Theorem 3.4.4. This follows from Lemmas 3.4.7 and 3.4.8. \square

Corollary 3.4.10. *Let $Y \subset \mathbb{R}^n$ be a set defined by a Boolean combination of definable functions $\{f_1, \dots, f_s\}$. Then there exists a closed set $X \subset \mathbb{R}^n$ defined by $2s^2 + 1$ definable functions with complexities $c(f_i)$, and $c(|x|)$ such that $X \simeq Y$.*

Theorem 3.4.11. *Let $Y \subset \mathbb{R}^n$ be a set defined by a Boolean combination of definable functions $\{f_1, \dots, f_s\}$. Then*

$$b(Y) \leq \sum_{i=0}^n \sum_{j=1}^{n-i} \binom{2s^2 + 1}{j} 6^j \Omega(F', \emptyset)$$

where F' contains all the elements f_i and $|x|^2$.

Example 3.4.12 (The Degree of a Polynomial). *We have $\Omega(F', \emptyset) = d(2d - 1)^{n-1}$, and we obtain Theorem 7.50 of the second edition of [2].*

Example 3.4.13 (Pfaffian Functions). *We have*

$$\Omega(F', \emptyset) = 2^{r(r-1)/2} \beta(\alpha + 2\beta - 1)^{n-1} (\min\{n, r\} \alpha + 2n\beta + (n-1)\alpha - 2n + 2)^r,$$

and we get

$$\begin{aligned} b(Y) &\leq \sum_{i=0}^n \sum_{j=1}^{n-i} \binom{2s^2 + 1}{j} 6^j 2^{r(r-1)/2} \beta(\alpha + 2\beta - 1)^{n-1} \\ &\quad \cdot (\min\{n, r\} \alpha + 2n\beta + (n-1)\alpha - 2n + 2)^r. \end{aligned}$$

Corollary 3.4.14.

$$\begin{aligned} b(Y) &\leq O(s^{2n} \Omega(F', \emptyset)) \\ &\leq O\left(s^{2n} \gamma\left(n, c\left(\sum_i f_i^2 + |x|^2\right)\right)\right) \end{aligned}$$

Proof. As in 3.3.9. □

Example 3.4.15 (The Degree of a Polynomial). *We have*

$$\gamma\left(n, c\left(\sum_i f_i^2 + |x|^2\right)\right) = 2d(2d - 1)^{n-1},$$

and we get the bound $O((s^2 d)^n)$.

Example 3.4.16 (Pfaffian Functions). *We get a bound*

$$\begin{aligned} b(X) &\leq O\left(s^{2n} 2^{r(r-1)/2} \beta(\alpha + 2\beta - 1)^{n-1} (\min\{n, r\} \alpha + 2n\beta + (n-1)\alpha - 2n + 2)^r\right) \\ &\leq O\left(s^{2n} 2^{r(r-1)/2} (\alpha + 2\beta)^n (\min\{n, r\} \alpha + 2n\beta + (n-1)\alpha - 2n + 2)^r\right) \\ &\leq O(s^{2n} 2^{r(r-1)/2} (2n\beta + (\min\{n, r\} + n - 1)\alpha)^{n+r}), \end{aligned}$$

which is equivalent to the bound stated in [22].

Chapter 4

Sets Defined with Quantifiers

4.1 Introduction

We now move on to tackle the most general situation, where X is a subset in $[-1, 1]^{n_0} \subset \mathbb{R}^{n_0}$ defined by

$$X = \{x_0 : Q_1 x_1 Q_2 x_2 \dots Q_\nu x_\nu ((x_0, x_1, \dots, x_\nu) \in X_\nu)\}, \quad (4.1.1)$$

where $Q_i \in \{\exists, \forall\}$, $Q_i \neq Q_{i+1}$, $x_i \in \mathbb{R}^{n_i}$, and X_ν is some definable set in $[-1, 1]^{n_0 + \dots + n_\nu}$.

Example 4.1.1. *If $\nu = 1$ and $Q_1 = \exists$, then X is the projection of X_1 .*

We make use of existing methods and ideas from two main sources in our new setting, as outlined below.

In [12], the authors present a method for bounding the Betti numbers of a surjective map using a particular spectral sequence. They then proceed to demonstrate how to bound the Betti numbers of a set of the form (4.1.1), in the case where X_ν is open or closed and is the difference between a finite CW-complex and its subcomplex. To achieve this, they firstly state forms of well-known results such as Alexander's Duality (which considers complements), the Mayer-Vietoris Inequality (which deals with unions and intersections), and De Morgan's Law. They also introduce a form of notation to manipulate various specific unions/intersections. These ideas are then used, along with the fact that the existential quantifier is equivalent to projection, to transform the problem into a form that can be solved using the spectral sequence method applied to the projection map. Finally, upper bounds for sub-Pfaffian sets are found by simply counting the number of terms in the expression resulting from the above, and directly calculating the complexity of the summand.

In [11], a method is shown to approximate a given definable set by a compact family T , based on sets that “represent” the given set. This “representation” is axiomatically defined, and a particular case of “representation” is constructed in the setting of a bounded definable set of points satisfying a Boolean combination of equations and inequalities. This result is then used to demonstrate how to improve existing bounds on Betti numbers of sub-Pfaffian sets,

both in the quantifier-free case, and, making use of the spectral sequence result above, in the case of a single projection.

My work seeks to combine the above ideas to produce a bound for any definable set of the form (4.1.1), in terms of the general complexity definition I give at the beginning of this work. I proceed by presenting various parts of the two above papers, adapted to my new context, and generalised to give the desired result.

Note that early results for semi-Pfaffian sets defined by quantifier-free formulae (see [22]) required the set to be restricted, i.e. $\text{closure}(X) \subset (0, 1)^n$; this requirement was removed in [11]. Similarly, the method shown in [12] requires the quantifier-free part of the set to be open or closed. Because our result uses the construction of a compact representation as above, which guarantees that the closed set $T \subset (0, 1)^n$, we can remove this restriction and produce a result for any set definable in an o-minimal structure over the reals, in particular, including those defined using quantifiers.

4.2 Sidenote – the case where X_ν is open or closed

In [12] a bound is given in terms of a summation of sets (an analogue to Theorem 4.9.8) in the case where X_ν is open or closed, and is the difference between a finite CW-complex and one of its subcomplexes. Our new axiomatic complexity measure can be applied to this bound to produce an explicit bound for sets defined in o-minimal structures. It is a simple calculation to show that this bound reduces to the results given in [12] in the polynomial and Pfaffian cases.

In the following we produce bounds that are weaker than the above. This is due to the fact that X_ν need not be open or closed, and not due to the new complexity metric.

4.3 A Spectral Sequence Associated with a Surjective Map

We now describe how to associate a particular mathematical object with a given surjective map; this in particular allows us to bound the Betti numbers of the codomain of this map. We will use this result later in the instance of projection maps. The mathematical object used is a spectral sequence; a good introduction to spectral sequence can be found in [5].

This section is taken directly from [12].

Definition 4.3.1. A continuous map $f : X \rightarrow Y$ is *locally split* if for any $y \in Y$ there is an open neighbourhood U of y and a section $s : U \rightarrow X$ of f (i.e., s is continuous and $fs = \text{Id}$). In particular, a projection of an open set in \mathbb{R}^n on a subspace of \mathbb{R}^n is always locally split.

Definition 4.3.2. For two maps $f_1 : X_1 \rightarrow Y$ and $f_2 : X_2 \rightarrow Y$, the *fibred product* of X_1 and X_2 is defined as

$$X_1 \times_Y X_2 = \{(x_1, x_2) \in X_1 \times X_2 : f_1(x_1) = f_2(x_2)\}.$$

Theorem 4.3.3. Let $f : X \rightarrow Y$ be a surjective cellular map. Assume that f is either closed or locally split. Then for any Abelian group G , there exists a spectral sequence $E_{p,q}^r$ converging

to $H_*(Y, G)$ with

$$E_{p,q}^1 = H_q(W_p, G)$$

where

$$W_p = \underbrace{X \times_Y \dots \times_Y X}_{p+1 \text{ times}}.$$

In particular,

$$\dim H_k(Y, G) \leq \sum_{p+q=k} \dim H_q(W_p, G),$$

for all k .

Proof. See [12], Theorem 1. □

4.4 Alexander's duality

Again the following is taken from [12].

Let

$$I_i^n = \bigcap_{1 \leq j \leq n} \{-i \leq x_j \leq i\} \subset \mathbb{R}^n.$$

Define the “thick boundary” $\partial I_i^n = I_{i+1}^n \setminus I_i^n$. The following is a version of Alexander's duality theorem.

Lemma 4.4.1 (Alexander's duality). *If $X \subset I_i^n$ is an open set in I_i^n , then for any $q \in \mathbb{Z}$, $q \leq n-1$,*

$$H_q(I_i^n \setminus X, \mathbb{R}) \cong \tilde{H}_{n-q-1}(X \cup \partial I_i^n, \mathbb{R}). \quad (4.4.1)$$

If $X \subset I_i^n$ is a closed set in I_i^n , then for any $q \in \mathbb{Z}$, $q \leq n-1$,

$$H_q(I_i^n \setminus X, \mathbb{R}) \cong \tilde{H}_{n-q-1}(X \cup \text{closure}(\partial I_i^n), \mathbb{R}). \quad (4.4.2)$$

Proof. See [12], Lemma 4. □

4.5 Approximation of Definable Sets by Compact Families

We now outline the methods shown in [11] to find a compact set to approximate an arbitrary set, which can belong to a large class of definable sets, including sets defined by any Boolean combination of equations and inequalities.

We take the following Definition and two Lemmas directly from [11]:

Definition 4.5.1. Let G be a definable compact set. Consider a definable family $\{S_\delta\}_{\delta>0}$ of compact subsets of G such that, for all $\delta', \delta \in (0, 1)$, if $\delta' > \delta$, then $S_{\delta'} \subset S_\delta$. Denote $S := \bigcup_{\delta>0} S_\delta$.

For each $\delta > 0$, let $\{S_{\delta,\varepsilon}\}_{\delta,\varepsilon>0}$ be a definable family of compact subsets of G such that:

- (i) for all $\varepsilon, \varepsilon' \in (0, 1)$, if $\varepsilon' > \varepsilon$, then $S_{\delta,\varepsilon} \subset S_{\delta,\varepsilon'}$;
- (ii) $S_\delta = \bigcap_{\varepsilon>0} S_{\delta,\varepsilon}$;
- (iii) for all $\delta' > 0$ sufficiently smaller than δ , and for all $\varepsilon' > 0$, there exists an open in G set $U \subset G$ such that $S_\delta \subset U \subset S_{\delta',\varepsilon'}$.

We say that S is *represented* by the families $\{S_\delta\}_{\delta>0}$ and $\{S_{\delta,\varepsilon}\}_{\delta,\varepsilon>0}$ in G .

Let S' be represented by $\{S'_\delta\}_{\delta>0}$ and $\{S'_{\delta,\varepsilon}\}_{\delta,\varepsilon>0}$ in G , and let S'' be represented by $\{S''_\delta\}_{\delta>0}$ and $\{S''_{\delta,\varepsilon}\}_{\delta,\varepsilon>0}$ in G .

Lemma 4.5.2. *The set $S' \cap S''$ is represented by the families $\{S'_\delta \cap S''_\delta\}_{\delta>0}$ and $\{S'_{\delta,\varepsilon} \cap S''_{\delta,\varepsilon}\}_{\delta,\varepsilon>0}$ in G , while $S' \cup S''$ is represented by $\{S'_\delta \cup S''_\delta\}_{\delta>0}$ and $\{S'_{\delta,\varepsilon} \cup S''_{\delta,\varepsilon}\}_{\delta,\varepsilon>0}$ in G .*

Proof. By checking of Definition 4.5.1. □

Let S be represented by $\{S_\delta\}_{\delta>0}$ and $\{S_{\delta,\varepsilon}\}_{\delta,\varepsilon>0}$ in G , and let $F : D \rightarrow H$ be a continuous definable map, where D and H are definable, $S \subset D \subset G$, and H is compact.

Lemma 4.5.3. *Let D be open in G , and F be an open map. Then $F(S)$ is represented by the families $\{F(S_\delta)\}_{\delta>0}$ and $\{F(S_{\delta,\varepsilon})\}_{\delta,\varepsilon>0}$ in H .*

Proof. Follows from checking Definition 4.5.1; openness is required for (iii) to hold. □

In [11], this general case is called the *definable* case (i.e. it has been axiomatically defined). A more specific case is actually constructed (in [11]) in the situation of a set defined by a Boolean combination of equations and inequalities, and is referred to as the *constructible* case. We outline this now, again taking the following directly from [11].

Let $S = \{x : \mathcal{F}(x)\} \subset \mathbb{R}^n$ be a bounded definable set of points satisfying a Boolean combination \mathcal{F} of equations of the kind $h(x) = 0$ and inequalities of the kind $h(x) > 0$, where $h : \mathbb{R}^n \rightarrow \mathbb{R}$ are continuous definable functions (e.g. polynomials). As G take a closed ball of sufficiently large radius centred at 0. We now define the representing families $\{S_\delta\}$ and $\{S_{\delta,\varepsilon}\}$.

Definition 4.5.4. For a given finite set $\{h_1, \dots, h_k\}$ of functions $h_i : \mathbb{R}^n \rightarrow \mathbb{R}$, define its *sign set* as a non-empty subset in \mathbb{R}^n of the kind

$$h_{i_1} = \dots = h_{i_{k_1}} = 0, h_{i_{k_1+1}} > 0, \dots, h_{i_{k_2}} > 0, h_{i_{k_2+1}} < 0, \dots, h_{i_k} < 0,$$

where $i_1, \dots, i_{k_1}, \dots, i_{k_2}, \dots, i_k$ is a permutation of $1, \dots, k$.

Let now $\{h_1, \dots, h_k\}$ be the set of all functions in the Boolean formula defining S . Then S is a disjoint union of some sign sets of $\{h_1, \dots, h_k\}$. The set S_δ is the result of the replacement, independently in each sign set in this union, of all inequalities $h > 0$ and $h < 0$ by $h \geq \delta$ and $h \leq -\delta$ respectively. The set $S_{\delta,\varepsilon}$ is obtained by replacing all expressions $h > 0$, $h < 0$, and

$h = 0$ by $h \geq \delta$, $h \leq -\delta$ and $-\varepsilon \leq h \leq \varepsilon$ respectively. According to Lemma 4.5.2, the set S being the union of sign sets, is represented by the families $\{S_\delta\}$ and $\{S_{\delta,\varepsilon}\}$ in G .

Now suppose that the set $S \subset \mathbb{R}^n$, defined as above by a Boolean formula \mathcal{F} , is not necessarily bounded. In this case as G take the definable one-point (Alexandrov) compactification of \mathbb{R}^n . Note that each function h is continuous in $G \setminus \{\infty\}$. Define sets S_δ and $S_{\delta,\varepsilon}$ as in the bounded case, replacing equations and inequalities, independently in each sign set of $\{h_1, \dots, h_k\}$, and then taking the conjunction of the resulting formula with $|x|^2 \leq 1/\delta$. Again, S is represented by $\{S_\delta\}$ and $\{S_{\delta,\varepsilon}\}$ in G .

We now return to the more general, *definable*, case. The following Definitions and Lemma are taken from [11], with a slight modification of notation to indicate that T depends on the family chosen to represent S , and on the value m (as defined below).

Definition 4.5.5. For a sequence $\varepsilon_0, \delta_0, \varepsilon_1, \delta_1, \dots, \varepsilon_m, \delta_m$, where $m \geq 0$, and a particular representation $\Upsilon = \{S_{\delta,\varepsilon}\}_{\delta,\varepsilon > 0}$ of S , introduce the compact set

$$T_{\Upsilon,m}(S) := S_{\delta_0,\varepsilon_0} \cup S_{\delta_1,\varepsilon_1} \cup \dots \cup S_{\delta_m,\varepsilon_m}.$$

In the following, see Definition 1.5.22 for the \ll symbol.

Lemma 4.5.6. For any $m \geq 0$, for

$$0 < \varepsilon_0 \ll \delta_0 \ll \varepsilon_1 \ll \delta_1 \ll \dots \ll \varepsilon_m \ll \delta_m \ll 1 \quad (4.5.1)$$

and for a representation Υ of S , there is a surjective map $C : \mathbf{T} \rightarrow \mathbf{S}$ from the finite set \mathbf{T} of all connected components of $T_{\Upsilon,m}(S)$ onto the set \mathbf{S} of all the connected components of S such that, for any $S' \in \mathbf{S}$, we have

$$\bigcup_{T' \in C^{-1}(S')} T' = T_{\Upsilon,m}(S').$$

If $m > 0$, then C is bijective.

Proof. The surjectivity follows directly from Definition 4.5.1. Bijectivity is proven in Lemma 1.4 of [11]. \square

The previous Lemma essentially states that the passage to $T_{\Upsilon,m}$ preserves connectivity. The next Theorem is essentially the key result in [11], and is again taken directly.

In the remained of this document, we denote $T := T_{\Upsilon,m}(S)$ whenever this does not lead to confusion.

We assume $m > 0$, and that S is connected in order to make the homotopy groups $\pi_k(S)$ and $\pi_k(T)$ independent of a base point.

Theorem 4.5.7. For a given representation Υ of S :

(i) For (4.5.1) and for every $1 \leq k \leq m$, there are epimorphisms

$$\begin{aligned}\psi_k &: \pi_k(T) \rightarrow \pi_k(S), \\ \phi_k &: H_k(T) \rightarrow H_k(S),\end{aligned}$$

and, in particular, $\text{rank } H_k(S) \leq \text{rank } H_k(T)$.

(ii) In the constructible case, for (4.5.1) and for every $1 \leq k \leq m - 1$, ψ_k and ϕ_k are isomorphisms, in particular $\text{rank } H_k(S) = \text{rank } H_k(T)$. Moreover, if $m \geq \dim(S)$, then $T \simeq S$.

Proof. See [11], Theorem 1.5. □

Note that in the above Theorem π_k refers to homotopy groups, not projections. The following Corollary is the direct implication of this existing work in our new setting, and is the only original item in this section.

Corollary 4.5.8. *Let $S \subset \mathbb{R}^n$ be defined by an arbitrary Boolean combination of s distinct definable functions h_i , with maximum complexity $\max(c(h_i))$. Then there exists a compact set $T \subset \mathbb{R}^n$ defined by a Boolean combination (with no negations) of $4(n + 1)s$ non-strict inequalities, with maximum complexity $\max(c(h_i), c(|x|^2))$, with $T \simeq S$.*

4.5.1 Projections

To move towards dealing with quantifiers, we take a first look at how to deal with projections. This section is expanded on in more detail later, we only touch on it now to show how the previous work can be applied in our new setting. In the following definition, we show that $T(\pi(S))$ can indeed be defined in the natural way.

Definition 4.5.9. Let $\pi : \mathbb{R}^{n+r} \rightarrow \mathbb{R}^n$ be the projection function, and $S \subset \mathbb{R}^{n+r}$. The set S is represented by the families $\{S_\delta\}_\delta$ and $\{S_{\delta,\varepsilon}\}_{\delta,\varepsilon}$ in the compactification of \mathbb{R}^{n+r} as described above. According to Lemma 4.5.3, the projection $\pi(S)$ is represented by the families $\{\pi(S_\delta)\}_\delta$ and $\{\pi(S_{\delta,\varepsilon})\}_{\delta,\varepsilon}$ in the compactification of \mathbb{R}^n , we call this particular representation Π . Fix $m = n + r$. Then for a given representation Υ of S , we can define $\pi(T_{\Upsilon,m}(S))$ to be equal to $T_{\Pi,m}(\pi(S))$.

We now demonstrate how this can be applied in the constructible case to give a bound using our general complexity definition, analogous to bounds given in [11] for the polynomial (complexity = degree) case, and for the Pfaffian case. Let $S = \{(x, y) : \mathcal{F}(x, y)\} \subset \mathbb{R}^{n+r}$, where \mathcal{F} is a Boolean combination of definable equations and inequalities of the kind $h_i(x, y) = 0$ or $h_i(x, y) > 0$, where h_i are definable functions ($1 \leq i \leq s$). Suppose that the maximum complexity of these functions is $\max(c(h_i))$.

Lemma 4.5.10.

$$b_k(\pi(S)) \leq \left(\frac{k^3 + 4k^2 + 5k + 2}{2} s \right)^{n+(k+1)r} O \left(\sum_{0 \leq p \leq k} \gamma \left(n + (p+1)r, c \left(\sum_i h_i^2 + |x|^2 \right) \right) \right)$$

Proof. For $k = 0$, we have $b_0(\pi(S)) \leq b_0(S)$. Lemma 4.5.6 tells us that there exists a surjective map from the set of connected components of $T_{\Upsilon, m}(S)$ to the set of connected components of S , and therefore $b_0(S) \leq b_0(T_{\Upsilon, m}(S))$. The set $T_{\Upsilon, m}$ is defined by $4(k+1)s$ definable functions and is in \mathbb{R}^n , hence, by Corollary 3.3.9, $b_0(T_{\Upsilon, m}(S)) \leq O(s^n \gamma(n, c(\sum_i h_i^2 + |x|^2)))$.

Now assume $k > 0$, and fix $m = k$. From Definition 4.5.9, we have $T_{\Pi, m}(\pi(S)) = \pi(T_{\Upsilon, m}(S))$, and recall from Section 1.5.6 that $b_i(X) = \text{rank } H_i(X)$. According to Theorem 4.3.3, we have

$$b_k(\pi(T_{\Upsilon, m}(S))) \leq \sum_{p+q=k} b_q(W_p),$$

where

$$W_p = \underbrace{T_{\Upsilon, m}(S) \times_{\pi(T_{\Upsilon, m}(S))} \cdots \times_{\pi(T_{\Upsilon, m}(S))} T_{\Upsilon, m}(S)}_{p+1 \text{ times}}.$$

The fibre product $W_p \subset \mathbb{R}^{n+(p+1)r}$ is definable by a Boolean formula with

$$4(p+1)(k+1)s$$

o-minimal functions. Hence, by Corollary 3.3.9, we have

$$b_q(W_p) \leq O \left(((p+1)(k+1)s)^{n+(p+1)r} \gamma(n + (p+1)r, c(\sum_i h_i^2 + |x|^2)) \right).$$

It follows that

$$\begin{aligned} b_k(T_{\Pi, m}(\pi(S))) &\leq \sum_{0 \leq p \leq k} O \left(((p+1)(k+1)s)^{n+(p+1)r} \gamma(n + (p+1)r, c(\sum_i h_i^2 + |x|^2)) \right) \\ &\leq \left(\frac{k^3 + 4k^2 + 5k + 2}{2} s \right)^{n+(k+1)r} O \left(\sum_{0 \leq p \leq k} \gamma(n + (p+1)r, c(\sum_i h_i^2 + |x|^2)) \right). \end{aligned}$$

Finally, by Theorem 4.5.7, $b_k(\pi(S)) \leq b_k(T_{\Pi, m}(\pi(S)))$, and the result follows. \square

Example 4.5.11 (The Degree of a Polynomial). *In the case of polynomials and degree, we have*

$$\gamma \left(n + (p+1)r, c \left(\sum_i h_i^2 + |x|^2 \right) \right) \leq O(d^{n+(p+1)r}),$$

and therefore $b_k(\pi(S)) \leq ((k+1)sd)^{O(n+kr)}$, which is the result given in [11] Theorem 6.4.

Example 4.5.12 (Pfaffian Functions). *In this case we have*

$$\gamma \left(n + (p+1)r, c \left(\sum_i h_i^2 + |x|^2 \right) \right) \leq O(2^{r^2} ((n + (p+1)r)(\alpha + \beta))^{n+(p+1)r})$$

and therefore

$$b_k(\pi(S)) \leq (ks)^{O(n+(k+1)r)} 2^{(kl)^2} ((n + (k+1)r)(\alpha + \beta))^{n+(k+1)r+kl},$$

which is the result given in [11] Theorem 6.8.

4.6 Notation

In the following, remember \overline{A} means the complement of A . Existing work ([12]) defines an initial set X , and proceeds to define sets X_i that fit the relation $\pi(X_i) = \overline{X_{i-1}}$, and uses this recurrence later in the inductive proof of the main result. However, in this setting we must define things slightly differently, as we are using the concept of “representation”, and we must be able to find a set to “represent” the set in the recurrence relation. This requires the map to be open, and the complement of the projection does not necessarily take open sets to open sets. We can circumvent this difficulty by using projection and co-projection (complement of the projection of the complement) in our definition, as they are both open maps (i.e. take open sets to open sets). We therefore use the slightly more cumbersome notation that follows, where later work will come to depend on the parity of i .

We assume that $X \subset [-1, 1]^{n_0} \subset \mathbb{R}^{n_0}$, and $X_\nu \subset [-1, 1]^{n_0+\dots+n_\nu}$. To restate our original formula, we have

$$X = \{x_0 : Q_1 x_1 Q_2 x_2 \dots Q_\nu x_\nu ((x_0, x_1, \dots, x_\nu) \in X_\nu)\}, \quad (4.6.1)$$

where $Q_i \neq Q_{i+1}$. We will also assume that $Q_1 = \exists$, the other case can be dealt with using Alexander’s Duality (and is explicitly dealt with later using another method in the case of exactly one universal quantifier).

Let $\overline{\pi(S)}$ be denoted by $\text{cp}(S)$. We now define sets

$$X_i = \{(x_0, \dots, x_i) : Q_{i+1} x_{i+1} Q_{i+2} x_{i+2} \dots Q_\nu x_\nu ((x_0, x_1, \dots, x_\nu) \in X_\nu)\}.$$

This definition satisfies

$$\begin{aligned}
X &= X_0 \\
&= \pi_1(X_1) \\
&= \pi_1(cp_2(X_2)) \\
&= \pi_1(cp_2(\pi_3(X_3))) \\
&\vdots \\
&= \pi_1(cp_2(\pi_3(\dots(\tau_i(X_i))))),
\end{aligned}$$

where $\tau_i = \pi_i$ if i is odd, and $\tau_i = cp_i$ if i is even. Essentially this notation gives identical sets to the existing notation for odd i , and the complement of previous sets for even i .

For a set $I_i^{m_i} \times I_{i-1}^{m_{i-1}} \times \dots \times I_1^{m_1}$, define $\partial(I_i^{m_i} \times I_{i-1}^{m_{i-1}} \times \dots \times I_1^{m_1})$ as

$$(I_{i+1}^{m_i} \times I_i^{m_{i-1}} \times \dots \times I_2^{m_1}) \setminus (I_i^{m_i} \times I_{i-1}^{m_{i-1}} \times \dots \times I_1^{m_1})$$

for even i , and as the closure of this difference for odd i .

Let p_1, \dots, p_i be some positive integers to be defined later. Define

$$B_i^i = \partial(I_{i-1}^{n_0+(p_1+1)n_1} \times I_{i-2}^{(p_2+1)n_2} \times \dots \times I_1^{(p_{i-1}+1)n_{i-1}}) \times I_1^{n_i}.$$

For any j , $i < j \leq \nu$, define $B_j^i = \overline{B_{j-1}^i} \times I_1^{n_j}$, where complements are in appropriate cubes.

4.6.1 General Intersections and Unions

We now define an “intersection” of particular types of sets, again taking all Definitions and Lemmas in this section directly from [12].

Definition 4.6.1. (i) Let $Y \subset I_v^{n_0} \times I_v^{(p_1+1)n_1} \times I_{v-1}^{(p_2+1)n_2} \times \dots \times I_{\nu-l+2}^{(p_{l-1}+1)n_{l-1}} \times I_1^{n_l+\dots+n_i}$, where $1 \leq l \leq i$, $v \geq i$, and let $J \subset \{(j_l, \dots, j_i) : 1 \leq j_k \leq p_k + 1, l \leq k \leq i\}$. Then define $\prod_{i,J}^l Y$ as an intersection of sets

$$\begin{aligned}
&\{(x_0, x_1^{(1)}, \dots, x_1^{(p_1+1)}, \dots, x_i^{(1)}, \dots, x_i^{(p_i+1)}) : \\
&\quad x_0 \in I_v^{n_0}, x_k^{(m)} \in I_{v-k+1}^{n_k} (1 \leq k \leq l-1), \\
&\quad x_k^{(m)} \in I_1^{n_k} (l \leq k \leq i), (x_0, x_1^{(1)}, \dots, x_{l-1}^{(p_{l-1}+1)}, x_l^{(j_l)}, \dots, x_i^{(j_i)}) \in Y\}
\end{aligned}$$

over all $(j_l, \dots, j_i) \in J$.

(ii) Let $Y \subset I_v^{n_0} \times I_v^{(p_1+1)n_1} \times I_{v-1}^{(p_2+1)n_2} \times \dots \times I_{\nu-l+2}^{(p_{l-1}+1)n_{l-1}} \times I_1^{n_l+\dots+n_i+n_{i+1}}$. Define $\prod_{i,J}^{l,i+1} Y$

as an intersection of sets

$$\begin{aligned} & \{(x_0, x_1^{(1)}, \dots, x_1^{(p_1+1)}, \dots, x_i^{(1)}, \dots, x_i^{(p_i+1)}, x_{i+1}) : \\ & \quad x_0 \in I_v^{n_0}, x_k^{(m)} \in I_{v-k+1}^{n_k} (1 \leq k \leq l-1), x_k^{(m)} \in I_1^{n_k} (l \leq k \leq i), \\ & \quad x_{i+1} \in I_1^{n_{i+1}}, (x_0, x_1^{(1)}, \dots, x_{l-1}^{(p_{l-1}+1)}, x_l^{(j_l)}, \dots, x_i^{(j_i)}) \in Y\} \end{aligned}$$

over all $(j_l, \dots, j_i) \in J$.

(iii) If $l = i$ and $J = \{j : 1 \leq j \leq p_i + 1\}$ we use the notation $\prod_i^i Y$ for $\prod_{i,J}^i Y$

The following shows how these intersections combine:

Lemma 4.6.2. *Let*

$$Y \subset I_v^{n_0} \times I_v^{(p_1+1)n_1} \times I_{v-1}^{(p_2+1)n_2} \times \dots \times I_{v-l+2}^{(p_{l-1}+1)n_{l-1}} \times I_1^{n_l+\dots+n_i+n_{i+1}}.$$

Then for any $J \subset \{j : 1 \leq j \leq p_{i+1} + 1\}$, $J' \subset \{(j_l, \dots, j_i) : 1 \leq j_k \leq p_k + 1, l \leq k \leq i\}$ we have

$$\prod_{i+1,J}^{i+1} \prod_{i,J'}^{l,i+1} Y = \prod_{i+1,J' \times J}^l Y.$$

Proof. Straightforward. □

We similarly define unions:

Definition 4.6.3. Let Y, l, i, J be as in Definition 4.6.1. Define $\bigsqcup_{i,J}^l Y$ and $\bigsqcup_{i,J}^{l,i+1} Y$ similar to $\prod_{i,J}^l Y$ and $\prod_{i,J}^{l,i+1} Y$ respectively, replacing in Definition 4.6.1 “intersection” by “union”.

We now state a version of De Morgan’s law, which show how unions, intersections, and complements interact.

Lemma 4.6.4 (De Morgan’s Law).

$$\begin{aligned} \bigsqcup_{i,J}^l Y &= \overline{\left(\prod_{i,J}^l \overline{Y} \right)} \\ \bigsqcup_{i,J}^{l,i+1} Y &= \overline{\left(\prod_{i,J}^{l,i+1} \overline{Y} \right)} \end{aligned}$$

where complements are in appropriate cubes.

Proof. Straightforward. □

We now define projections:

Definition 4.6.5. Let $t_i = n_0 + n_1(p_1 + 1) + \dots + n_i(p_i + 1)$. Define projection maps

$$\pi_i : \mathbb{R}^{n_0 + \dots + n_i} \rightarrow \mathbb{R}^{n_0 + \dots + n_{i-1}}$$

$$(x_0, \dots, x_i) \mapsto (x_0, \dots, x_{i-1})$$

and for $j < i$,

$$\pi_{i,j} : \mathbb{R}^{t_j + n_{j+1} + \dots + n_i} \rightarrow \mathbb{R}^{t_j + n_{j+1} + \dots + n_{i-1}}$$

$$(x_0, x_1^{(1)}, \dots, x_j^{(p_j+1)}, x_{j+1}, \dots, x_i) \mapsto (x_0, x_1^{(1)}, \dots, x_j^{(p_j+1)}, x_{j+1}, \dots, x_{i-1}).$$

We finally show how unions and projections commute

Lemma 4.6.6. *Let*

$$Y \subset I_v^{n_0} \times I_v^{(p_1+1)n_1} \times I_{v-1}^{(p_2+1)n_2} \times \dots \times I_{v-l+2}^{(p_{l-1}+1)n_{l-1}} \times I_1^{n_l + \dots + n_i + n_{i+1}}.$$

Then

$$\bigsqcup_{i,J}^l \pi_{i+1,l-1}(Y) = \pi_{i+1,i} \left(\bigsqcup_{i,J}^{l,i+1} Y \right).$$

Proof. Straightforward. □

4.7 Application of Spectral Sequence Result

We present a Lemma that will allow us to use Theorem 4.3.3 in our proof.

Lemma 4.7.1. *For any closed $A \subset I_v^{n_0} \times I_v^{(p_1+1)n_1} \times I_{v-1}^{(p_2+1)n_2} \times \dots \times I_{v-i+2}^{(p_{i-1}+1)n_{i-1}} \times I_1^{n_i}$,*

$$b_k(\pi_{i,i-1}(A)) \leq \sum_{p_i + q_i = k} b_{q_i} \left(\bigcap_i^i (A) \right).$$

Proof. Theorem 4.3.3 tells us that

$$b_k(Y) \leq \sum_{p+q=k} b_q(W_p).$$

We will take $Y = \pi_{i,i-1}(A)$, and $f = \pi_{i,i-1}$. For convenience, let us say $I = I_v^{n_0} \times I_v^{(p_1+1)n_1} \times I_{v-1}^{(p_2+1)n_2} \times \dots \times I_{v-i+2}^{(p_{i-1}+1)n_{i-1}}$ so $A \subset I \times I_1^{n_i}$, so we have $\pi_{i,i-1}(A) \subset I$.

From the definition,

$$W_{p_i} = \{((a, b_1), (a, b_2), \dots, (a, b_{p_i+1}) : a \in I, b_j \in I_1^{n_i}, (a, b_j) \in A, 1 \leq j \leq p_i + 1\}.$$

Similarly, we have

$$\bigcap_i^i (A) = \{(a, b_1, b_2, \dots, b_{p_i+1}) : a \in I, b_j \in I_1^{n_i}, (a, b_j) \in A, 1 \leq j \leq p_i + 1\},$$

and the result follows. □

4.8 Outline of Proof

We now have all the tools we need to resolve the main problems of this section. Our proof of the main result is through induction on the number of quantifiers. The case of one existential quantifier (equivalent to projection) is straightforward, and indeed is essentially done in Section 4.5.1. The one universal quantifier case can be deduced using Alexander’s duality, but we present another proof, which is a good introduction to more complex proofs to follow. The two and three quantifiers cases are also given, as these illustrate the method of proof more clearly than if one immediately moved to the general case. These two extra cases are only presented with $Q_1 = \exists$; the corresponding results for $Q_1 = \forall$ can either be deduced through similar methods, or directly from these results using Alexander’s duality.

Our main proof in this section, that of the case of an arbitrary number of quantifiers, makes use of most of the preceding results in this chapter. To summarise these results we have:

- Lemma 4.7.1 shows us how to use Theorem 4.3.3 to transform the problem of finding Betti numbers of projections into finding the sum of Betti numbers of “intersections” as defined in Definition 4.6.1
- Lemma 4.4.1, Alexander’s Duality, shows how to take the Betti numbers of complements.
- Lemma 1.5.23, the Mayer-Vietoris Inequality, shows how to turn intersections into unions, and unions into intersections.
- Lemma 4.6.2 shows us how to combine two of the defined “intersections” into one.
- Lemma 4.6.4, De Morgan’s Law, shows how the complement of the union is the intersection of the complements.
- Lemma 4.6.6 shows how unions and projections commute.
- Section 4.5 shows us how to approximate a given set by a compact set, provided we can find sets to “represent” it.

We use these tools, along with the fact that the existential quantifier is equivalent to projection, to reduce the problem of ν quantifiers into the problem of $\nu - 1$ quantifiers, and derive the result using induction.

4.9 Quantifiers

In the following, we assume that when calculating $T_{\Upsilon, m}(S)$ for any given set S , m is chosen with $m \geq \dim(S)$, which ensures $b_k(S) \leq b_k(T_{\Upsilon, m}(S))$ for every k .

4.9.1 One Existential Quantifier

Let us restate the original formula being studied, Equation (4.1.1):

$$X = \{x_0 : Q_1 x_1 Q_2 x_2 \dots Q_\nu x_\nu ((x_0, x_1, \dots, x_\nu) \in X_\nu)\}.$$

We will now consider the case where $\nu = 1$ and $Q_1 = \exists$, therefore

$$X = \{x_0 : \exists x_1 ((x_0, x_1) \in X_1)\},$$

where $X_1 = \{F(x_0, x_1)\}$. Let F be a Boolean combination of definable equations and inequalities of the kind $f_i = 0$ and $f_i > 0$, where there are s different functions f_i .

Lemma 4.9.1. *For any representation Υ of X_1 ,*

$$b_{q_0}(X) \leq \sum_{p_1+q_1=q_0} b_{q_1} \left(\prod_{1, J_1^1}^1 T_{\Upsilon, m}(X_1) \right),$$

where $J_1^1 = \{1, \dots, p_1 + 1\}$.

Proof. We have

$$b_{q_0}(X) = b_{q_0}(\pi_1(X_1)),$$

remembering that Π is the representation defined via the projection in Definition 4.5.9, from Theorem 4.5.7 the previous expression is bounded by

$$b_{q_0}(T_{\Pi, m}(\pi_1(X_1))),$$

which by Definition 4.5.9 is equal to

$$b_{q_0}(\pi_1(T_{\Upsilon, m}(X_1))),$$

and by Lemma 4.7.1 this is bounded by

$$\sum_{p_1+q_1=q_0} b_{q_1} \left(\prod_1^1 (T_{\Upsilon, m}(X_1)) \right).$$

□

4.9.2 One Universal Quantifier

We will now consider the case where $\nu = 1$ and $Q_1 = \forall$, therefore

$$X = \{x_0 : \forall x_1 ((x_0, x_1) \in X_1)\},$$

where $X_1 = \{F(x_0, x_1)\}$. Note that a single universal quantifier is equivalent to $\neg\exists\neg$, and therefore equivalent to the complement of the projection of the complement, or the “co-projection”.

This can be done relatively directly using Alexander’s Duality, but we present the following method as it is how we approach the same situation later.

In the following section, it is important to remember that we use \overline{A} to mean the complement of A , and that we let $\overline{\pi(\overline{S})}$ be denoted by $\text{cp}(S)$.

Lemma 4.9.2. *If $A \subset B$, then $\text{cp}(A) \subset \text{cp}(B)$.*

Proof. We have $A \subset B$, therefore $\overline{A} \supset \overline{B}$, $\pi(\overline{A}) \supset \pi(\overline{B})$, and $\overline{\pi(\overline{A})} \subset \overline{\pi(\overline{B})}$. □

When taking complements throughout the rest of this chapter, remember that $X_\nu \subset [-1, 1]^{n_0 + \dots + n_\nu}$.

The next Lemma follows from Lemma 4.5.3, but is explicitly proven here as it is vital in later work.

Lemma 4.9.3. *Let G be a definable compact set, and let $\{S_{\delta,\varepsilon}\}_{\delta,\varepsilon>0}$ be a definable family of compact subsets of G . If S is represented by $\{S_{\delta,\varepsilon}\}_{\delta,\varepsilon>0}$ then $\text{cp}(S)$ is represented by $\{\text{cp}(S_{\delta,\varepsilon})\}_{\delta,\varepsilon>0}$.*

Proof. We must show that the three requirements in the definition of “representation” (Definition 4.5.1) hold. For any $\delta > 0$ we have:

- (i) For all $\varepsilon, \varepsilon' \in (0, 1)$, if $\varepsilon' > \varepsilon$, then $S_{\delta,\varepsilon} \subset S_{\delta,\varepsilon'}$, and thus from Lemma 4.9.2 we have $\text{cp}(S_{\delta,\varepsilon}) \subset \text{cp}(S_{\delta,\varepsilon'})$.
- (ii) $S_\delta = \bigcap_{\varepsilon>0} S_{\delta,\varepsilon}$, so $\text{cp}(S_\delta) = \text{cp}(\bigcap_{\varepsilon>0} S_{\delta,\varepsilon}) = \bigcap_{\varepsilon>0} \text{cp}(S_{\delta,\varepsilon})$.
- (iii) Since S represented by $\{S_{\delta,\varepsilon}\}_{\delta,\varepsilon>0}$, for all $\delta' > 0$ sufficiently smaller than δ , and for all $\varepsilon' > 0$, there exists an open in G set $U \subset G$ such that $S_\delta \subset U \subset S_{\delta',\varepsilon'}$. We therefore have $\text{cp}(S_\delta) \subset \text{cp}(U) \subset \text{cp}(S_{\delta',\varepsilon'})$. Since $\text{cp}(U)$ is open if U is open, and indeed U is open, this condition is satisfied.

□

We now reformulate Lemma 4.7.1 for open sets:

Lemma 4.9.4. *For any open*

$$A \subset I_v^{n_0} \times I_v^{(p_1+1)n_1} \times I_{v-1}^{(p_2+1)n_2} \times \dots \times I_{v-i+2}^{(p_{i-1}+1)n_{i-1}} \times I_1^{n_i},$$

$$b_k(\pi_{i,i-1}(A)) \leq \sum_{p_i+q_i=k} b_{q_i}(\bigcap_i^i(A)).$$

This is used in the following, where we consider the case where $\nu = 1$ and $Q_1 = \forall$:

Lemma 4.9.5.

$$b_{q_0}(X) = b_{q_0}(\overline{\pi_1(\overline{(X_1)})})$$

$$\leq \sum_{J \subset \{1, \dots, m+1\}} \sum_{p_1+q_1=n_0-q_0-2+|J|} b_{q_1} \left(\prod_{1, J_1^1}^1 \left(\bigcup_{j \in J} \overline{(X_1)_{\delta_j, \varepsilon_j}} \cup \partial I_1^{n_0} \times I_1^{n_1} \right) \right).$$

Proof. Let Υ be the representation $\{\text{cp}((X_1)_{\delta, \varepsilon})\}_{\delta, \varepsilon > 0}$ of $\text{cp}(X_1)$, which is a valid representation from Lemma 4.9.3. We have from Theorem 4.5.7, $b_{q_0}(\text{cp}(X_1)) \leq b_{q_0}(T_{\Upsilon, m}(\text{cp}(X_1)))$, and explicitly using this representation and the definition of T we have

$$\begin{aligned} b_{q_0}(T_{\Upsilon, m}(\text{cp}(X_1))) &= b_{q_0}(\text{cp}((X_1)_{\delta_0, \varepsilon_0})) \cup \dots \cup \text{cp}((X_1)_{\delta_m, \varepsilon_m}) \\ &= b_{q_0} \left(\overline{\pi((X_1)_{\delta_0, \varepsilon_0})} \cup \dots \cup \overline{\pi((X_1)_{\delta_m, \varepsilon_m})} \right) \\ &= b_{q_0} \left(\pi \left(\overline{(X_1)_{\delta_0, \varepsilon_0}} \right) \cap \dots \cap \pi \left(\overline{(X_1)_{\delta_m, \varepsilon_m}} \right) \right) \end{aligned}$$

Now we have $(X_1)_{\delta, \varepsilon}$ is closed, so $\overline{(X_1)_{\delta, \varepsilon}}$ is open, $\pi(\overline{(X_1)_{\delta, \varepsilon}})$ is open, and the intersection of any finite collection of open sets is open, therefore the set we are considering is closed. By Alexander's Duality we have,

$$\begin{aligned} &b_{q_0} \left(\overline{\pi \left(\overline{(X_1)_{\delta_0, \varepsilon_0}} \right) \cap \dots \cap \pi \left(\overline{(X_1)_{\delta_m, \varepsilon_m}} \right)} \right) \\ &= b_{n_0-q_0-1} \left(\left(\pi \left(\overline{(X_1)_{\delta_0, \varepsilon_0}} \right) \cap \dots \cap \pi \left(\overline{(X_1)_{\delta_m, \varepsilon_m}} \right) \right) \cup \partial I_1^{n_0} \right) \end{aligned}$$

and using the Mayer-Vietoris Inequalities this does not exceed

$$\begin{aligned} &\sum_{J \subset \{1, \dots, m+1\}} b_{n_0-q_0-2+|J|} \left(\bigcup_{j \in J} \pi \left(\overline{(X_1)_{\delta_j, \varepsilon_j}} \right) \cup \partial I_1^{n_0} \right) \\ &= \sum_{J \subset \{1, \dots, m+1\}} b_{n_0-q_0-2+|J|} \left(\bigcup_{j \in J} \pi \left(\overline{(X_1)_{\delta_j, \varepsilon_j}} \right) \cup \pi(\partial I_1^{n_0} \times I_1^{n_1}) \right) \\ &= \sum_{J \subset \{1, \dots, m+1\}} b_{n_0-q_0-2+|J|} \left(\pi \left(\bigcup_{j \in J} \overline{(X_1)_{\delta_j, \varepsilon_j}} \cup \partial I_1^{n_0} \times I_1^{n_1} \right) \right). \end{aligned}$$

Now using Lemma 4.9.4 this is bounded by

$$\sum_{J \subset \{1, \dots, m+1\}} \sum_{p_1+q_1=n_0-q_0-2+|J|} b_{q_1} \left(\prod_{1, J_1^1}^1 \left(\bigcup_{j \in J} \overline{(X_1)_{\delta_j, \varepsilon_j}} \cup \partial I_1^{n_0} \times I_1^{n_1} \right) \right).$$

□

4.9.3 Two Quantifiers - $\exists \forall$

Lemma 4.9.6.

$$\begin{aligned}
b_{q_0}(X) &= b_{q_0}(\pi_1(\text{cp}_2(X_2))) \\
&\leq \sum_{p_1+q_1=q_0} \sum_{\hat{J}_1^1 \subset J_1^1} \sum_{J_2^2 \subset \hat{J}_1^1} \sum_{K_1 \subset \{1, \dots, m\}} \sum_{p_2+q_2=t_1-q_1-|\hat{J}_1^1|-|K_1|-|J_2^2|-1} \\
&\quad b_{q_2} \left(\prod_2^2 \left(\bigsqcup_{1, J_2^2}^{1,2} \overline{(X_2)_{\delta_i, \varepsilon_i}} \cup B_2^2 \right) \right)
\end{aligned}$$

Proof. Firstly, from Lemma 4.9.1:

$$\begin{aligned}
b_{q_0}(X) &= b_{q_0}(\pi_1(\text{cp}_2(X_2))) \\
&\leq \sum_{p_1+q_1=q_0} b_{q_1} \left(\prod_{1, J_1^1}^1 (T(\text{cp}_2(X_2))) \right)
\end{aligned}$$

and because of Lemma 4.9.3 this is equal to

$$\begin{aligned}
&\sum_{p_1+q_1=q_0} b_{q_1} \left(\prod_{1, J_1^1}^1 (\text{cp}_2((X_2)_{\delta_0, \varepsilon_0}) \cup \dots \cup \text{cp}_2((X_2)_{\delta_m, \varepsilon_m})) \right) \\
&= \sum_{p_1+q_1=q_0} b_{q_1} \left(\prod_{1, J_1^1}^1 \left(\overline{\pi_2((X_2)_{\delta_0, \varepsilon_0}) \cap \dots \cap \pi_2((X_2)_{\delta_m, \varepsilon_m})} \right) \right) \\
&= \sum_{p_1+q_1=q_0} b_{q_1} \left(\prod_{1, J_1^1}^1 \left(\overline{\pi_2((X_2)_{\delta_0, \varepsilon_0}) \cap \dots \cap \pi_2((X_2)_{\delta_m, \varepsilon_m})} \right) \right)
\end{aligned}$$

which by Alexander's Duality is bounded by

$$\begin{aligned}
&\sum_{p_1+q_1=q_0} b_{t_1-q_1-1} \left(\bigsqcup_{1, J_1^1}^1 \left(\pi_2((X_2)_{\delta_0, \varepsilon_0}) \cap \dots \cap \pi_2((X_2)_{\delta_m, \varepsilon_m}) \right) \cup \partial I_1^{t_1} \right) \\
&= \sum_{p_1+q_1=q_0} b_{t_1-q_1-1} \left(\bigsqcup_{1, J_1^1}^1 \left(\bigcap_{0 \leq i \leq m} \pi_2((X_2)_{\delta_i, \varepsilon_i}) \right) \cup \partial I_1^{t_1} \right).
\end{aligned}$$

The Mayer-Vietoris Inequalities in turn bound this by

$$\sum_{p_1+q_1=q_0} \sum_{\hat{J}_1^1 \subset J_1^1} b_{t_1-q_1-|\hat{J}_1^1|} \left(\prod_{1, \hat{J}_1^1}^1 \left(\bigcap_{0 \leq i \leq m} \pi_2(\overline{(X_2)_{\delta_i, \varepsilon_i}}) \right) \cup \partial I_1^{t_1} \right),$$

again using the Mayer-Vietoris Inequalities we have

$$\begin{aligned} & b_{t_1-q_1-|\hat{J}_1^1|} \left(\prod_{1, \hat{J}_1^1}^1 \left(\bigcap_{0 \leq i \leq m} \pi_2(\overline{(X_2)_{\delta_i, \varepsilon_i}}) \right) \cup \partial I_1^{t_1} \right) \\ & \leq \sum_{J_2^2 \subset \hat{J}_1^1} \sum_{K_1 \subset \{1, \dots, m\}} b_{t_1-q_1-|\hat{J}_1^1|-|K_1|-|J_2^2|-1} \left(\bigsqcup_{1, J_2^2}^1 \bigcup_{i \in K_1} \pi_2(\overline{(X_2)_{\delta_i, \varepsilon_i}}) \cup \partial I_1^{t_1} \right) \\ & = \sum_{J_2^2 \subset \hat{J}_1^1} \sum_{K_1 \subset \{1, \dots, m\}} b_{t_1-q_1-|\hat{J}_1^1|-|K_1|-|J_2^2|-1} \left(\bigsqcup_{1, J_2^2}^1 \bigcup_{i \in K_1} \pi_2(\overline{(X_2)_{\delta_i, \varepsilon_i}}) \cup \pi_{2,1}(\partial I_1^{t_1} \times I_1^{n_2}) \right) \\ & = \sum_{J_2^2 \subset \hat{J}_1^1} \sum_{K_1 \subset \{1, \dots, m\}} b_{t_1-q_1-|\hat{J}_1^1|-|K_1|-|J_2^2|-1} \left(\pi_{2,1} \left(\bigsqcup_{1, J_2^2}^{1,2} \bigcup_{i \in K_1} \overline{(X_2)_{\delta_i, \varepsilon_i}} \cup \partial I_1^{t_1} \times I_1^{n_2} \right) \right) \end{aligned}$$

and by Lemma 4.7.1 we have

$$\begin{aligned} & b_{t_1-q_1-|\hat{J}_1^1|-|K_1|-|J_2^2|-1} \left(\pi_{2,1} \left(\bigsqcup_{1, J_2^2}^{1,2} \bigcup_{i \in K_1} \overline{(X_2)_{\delta_i, \varepsilon_i}} \cup \partial I_1^{t_1} \times I_1^{n_2} \right) \right) \\ & \leq \sum_{p_2+q_2=t_1-q_1-|\hat{J}_1^1|-|K_1|-|J_2^2|-1} b_{q_2} \left(\prod_2^2 \left(\bigsqcup_{1, J_2^2}^{1,2} \bigcup_{i \in K_1} \overline{(X_2)_{\delta_i, \varepsilon_i}} \cup \partial I_1^{t_1} \times I_1^{n_2} \right) \right) \\ & = \sum_{p_2+q_2=t_1-q_1-|\hat{J}_1^1|-|K_1|-|J_2^2|-1} b_{q_2} \left(\prod_2^2 \left(\bigsqcup_{1, J_2^2}^{1,2} \bigcup_{i \in K_1} \overline{(X_2)_{\delta_i, \varepsilon_i}} \cup B_2^2 \right) \right) \end{aligned}$$

□

4.9.4 Three Quantifiers - $\exists \forall \exists$

Lemma 4.9.7.

$$\begin{aligned}
b_{q_0}(X) &= b_{q_0}(\pi_1(\text{cp}_2(\pi_3(X_3)))) \\
&\leq \sum_{p_1+q_1=q_0} \sum_{\hat{J}_1^1 \subset J_1^1} \sum_{J_2^2 \subset \hat{J}_1^1} \sum_{K_1 \subset \{1, \dots, m\}} \sum_{p_2+q_2=t_1-q_1-|\hat{J}_1^1|-|K_1|-|J_2^2|-1} \\
&\quad \sum_{1 \leq k_2 \leq p_2+1} \sum_{\hat{J}_2^2 \subset \{1, \dots, p_2+1\}, |\hat{J}_2^2|=k_2} \sum_{1 \leq s_2 \leq q_2+k_2+1+|K_1|} \\
&\quad \sum_{J_2^1 \subset J_1^1 \times \hat{J}_2^2, J_2^2 \subset J_1^2 \times \hat{J}_2^2, K_2 \subset K_1, |J_2^1|+|J_2^2|+|K_1|=s_2} \\
&\quad \sum_{p_3+q_3=t_2-q_2-k_2+s_2-1} b_{q_3} \left(\prod_3^3 \left(\prod_{2, J_2^1}^{1,3} \left(\bigcup_{i \in K_2} (X_3)_{\delta_i, \varepsilon_i} \right) \cup \prod_{2, J_2^2}^{2,3} B_3^2 \cup B_3^3 \right) \right)
\end{aligned}$$

where $J_1^1 = \{1, \dots, p_1 + 1\}$.

Proof. Working from the results in the previous section, we have a summand involving

$$b_{q_2} \left(\prod_2^2 \left(\prod_{1, J_2^2}^{1,2} \bigcup_{i \in K_1} \overline{(X_2)_{\delta_i, \varepsilon_i}} \cup B_2^2 \right) \right).$$

From De Morgan's Law and Alexander's duality, this is equal to

$$\begin{aligned}
&b_{q_2} \left(\overline{\prod_2^2 \left(\prod_{1, J_2^2}^{1,2} \bigcap_{i \in K_1} (X_2)_{\delta_i, \varepsilon_i} \cap \overline{B_2^2} \right)} \right) \\
&= b_{t_2-q_2-1} \left(\prod_2^2 \left(\prod_{1, J_2^2}^{1,2} \bigcap_{i \in K_1} (X_2)_{\delta_i, \varepsilon_i} \cap \overline{B_2^2} \right) \cup \partial(I_2^{t_2} \times I_1^{n_2(p_2+1)}) \right).
\end{aligned}$$

By the Mayer-Vietoris inequality the preceding is bounded by

$$\sum_{1 \leq k_2 \leq p_2+1} \sum_{\hat{J}_2^2 \subset \{1, \dots, p_2+1\}, |\hat{J}_2^2|=k_2} b_{t_2-q_2-k_2} \left(\prod_{2, \hat{J}_2^2}^2 \left(\prod_{1, J_2^2}^{1,2} \bigcap_{i \in K_1} (X_2)_{\delta_i, \varepsilon_i} \cap \overline{B_2^2} \right) \cup \partial(I_2^{t_2} \times I_1^{n_2(p_2+1)}) \right),$$

with the summand of this expression being equal to

$$b_{t_2-q_2-k_2} \left(\prod_{2, J_1^1 \times \hat{J}_2^2}^1 \left(\bigcap_{i \in K_1} (X_2)_{\delta_i, \varepsilon_i} \right) \cap \prod_{2, J_1^1 \times \hat{J}_2^2}^2 \overline{B_2^2} \cup \partial(I_2^{t_2} \times I_1^{n_2(p_2+1)}) \right),$$

with $J_1^2 = \{1\}$. By the Mayer-Vietoris inequality, this is bounded by

$$\sum_{1 \leq s_2 \leq q_2 + k_2 + 1 + |K_1|} \sum_{J_2^1 \subset J_1^1 \times \widehat{J}_2^2, J_2^2 \subset J_1^2 \times \widehat{J}_2^2, K_2 \subset K_1, |J_2^1| + |J_2^2| + |K_1| = s_2} b_{t_2 - q_2 - k_2 + s_2 - 1} \left(\bigsqcup_{2, J_2^1}^1 \left(\bigcup_{i \in K_2} (X_2)_{\delta_i, \varepsilon_i} \right) \cup \bigsqcup_{2, J_2^2}^2 \overline{B_2^2} \cup \partial(I_2^{t_2} \times I_1^{n_2(p_2+1)}) \right),$$

We have $X_2 = \pi_3(X_3)$, and $\pi_3(X_3)$ is represented by $\pi_3((X_3)_{\delta, \varepsilon})$, so the summand of the previous expression is equal to

$$b_{t_2 - q_2 - k_2 + s_2 - 1} \left(\bigsqcup_{2, J_2^1}^1 \left(\bigcup_{i \in K_2} \pi_3((X_3)_{\delta_i, \varepsilon_i}) \right) \cup \bigsqcup_{2, J_2^2}^2 \pi_{3,1}(B_3^2) \cup \pi_{3,2}(\partial(I_2^{t_2} \times I_1^{n_2(p_2+1)}) \times I_1^{n_3}) \right)$$

which is equal to

$$b_{t_2 - q_2 - k_2 + s_2 - 1} \left(\pi_{3,2} \left(\bigsqcup_{2, J_2^1}^{1,3} \left(\bigcup_{i \in K_2} (X_3)_{\delta_i, \varepsilon_i} \right) \cup \bigsqcup_{2, J_2^2}^{2,3} B_3^2 \cup B_3^3 \right) \right),$$

which in turn is bounded by

$$\sum_{p_3 + q_3 = t_2 - q_2 - k_2 + s_2 - 1} b_{q_3} \left(\bigsqcup_3^3 \left(\bigsqcup_{2, J_2^1}^{1,3} \left(\bigcup_{i \in K_2} (X_3)_{\delta_i, \varepsilon_i} \right) \cup \bigsqcup_{2, J_2^2}^{2,3} B_3^2 \cup B_3^3 \right) \right),$$

□

4.9.5 Arbitrary number of Quantifiers

In the following, we take $Q_1 = \exists$. The other case can be computed similarly, or calculated directly using Alexander's duality. We follow the general idea used in the two and three quantifier cases, and proceed using induction.

Theorem 4.9.8 (Main Result A). *For any i , let $\tau_i = \pi_i$ if i is odd and $\tau_i = cp_i$ if i is even.*

Let $\widehat{\cdot}$ denote complement if i is even, and have no effect if i is odd. Then for any i :

$$\begin{aligned}
b_{q_0}(X) &\leq b_{q_0}(\pi_1(cp_2(\pi_3(\dots(\tau_i(X_i))\dots)))) \\
&\leq \sum_{p_1+q_1=q_0} \sum_{\hat{J}_1^1 \subset J_1^1} \sum_{J_2^2 \subset \hat{J}_1^1} \sum_{K_1 \subset \{1, \dots, m\}} \\
&\quad \sum_{p_2+q_2=t_1-q_1-|\hat{J}_1^1|-|K_1|-|J_2^2|-1} \sum_{1 \leq k_2 \leq p_2+1} \sum_{\hat{J}_2^2 \subset \{1, \dots, p_2+1\}, |\hat{J}_2^2|=k_2} \\
&\quad \sum_{1 \leq s_2 \leq q_2+k_2+1+|K_1|} \sum_{J_2^1 \subset J_1^1 \times \hat{J}_2^2, J_2^2 \subset J_1^2 \times \hat{J}_2^2, |J_2^1|+|J_2^2|=s_2} \sum_{p_3+q_3=t_2-q_2-k_2+s_2-1} \dots \\
&\quad \dots \sum_{1 \leq k_{i-1} \leq p_{i-1}+1} \sum_{\hat{J}_{i-1}^{i-1} \subset \{1, \dots, p_{i-1}+1\}, |\hat{J}_{i-1}^{i-1}|=k_{i-1}} \sum_{1 \leq s_{i-1} \leq q_{i-1}+k_{i-1}+1+|K_{i-2}|} \\
&\quad \sum_{J_{i-1}^1 \subset J_{i-2}^1 \times \hat{J}_{i-1}^{i-1}, \dots, J_{i-1}^{i-1} \subset J_{i-2}^{i-1} \times \hat{J}_{i-1}^{i-1}, K_{i-1} \subset K_{i-2}, |J_{i-1}^1| + \dots + |J_{i-1}^{i-1}| + |K_{i-1}| = s_{i-1}} \\
&\quad \sum_{p_i+q_i=t_{i-1}-q_{i-1}-k_{i-1}+s_{i-1}-1} \\
&\quad b_{q_i} \left(\bigcap_i \left(\bigcup_{i-1, J_{i-1}^1}^{1, i} \left(\bigcup_{j \in K_{i-1}} \widehat{(X_i)_{\delta_j, \varepsilon_j}} \right) \cup \bigcup_{2 \leq r \leq i-1} \bigcup_{i-1, J_{i-1}^r}^{r, i} B_i^r \cup B_i^i \right) \right),
\end{aligned} \tag{4.9.1}$$

where $J_1^1 = \{1, \dots, p_1 + 1\}$.

Proof. By induction on i . We have dealt with the cases $i = 1, 2, 3$ above. Now, suppose Equation (4.9.1) is true, and i is odd. Then, from De Morgan's Law (Lemma 4.6.4) and Alexander's Duality (Lemma 4.4.1) we have:

$$\begin{aligned}
&b_{q_i} \left(\bigcap_i \left(\bigcup_{i-1, J_{i-1}^1}^{1, i} \left(\bigcup_{j \in K_{i-1}} (X_i)_{\delta_j, \varepsilon_j} \right) \cup \bigcup_{2 \leq r \leq i-1} \bigcup_{i-1, J_{i-1}^r}^{r, i} B_i^r \cup B_i^i \right) \right) \\
&= b_{q_i} \left(\overline{\left(\bigcup_i \left(\bigcap_{i-1, J_{i-1}^1}^{1, i} \left(\bigcap_{j \in K_{i-1}} \overline{(X_i)_{\delta_j, \varepsilon_j}} \right) \cap \bigcap_{2 \leq r \leq i-1} \bigcap_{i-1, J_{i-1}^r}^{r, i} \overline{B_i^r} \cup \overline{B_i^i} \right) \right)} \right) \\
&\leq b_{t_i-q_i-1} \left(\bigcup_i \left(\bigcap_{i-1, J_{i-1}^1}^{1, i} \left(\bigcap_{j \in K_{i-1}} \overline{(X_i)_{\delta_j, \varepsilon_j}} \right) \cap \bigcap_{2 \leq r \leq i-1} \bigcap_{i-1, J_{i-1}^r}^{r, i} \overline{B_i^r} \cap \overline{B_i^i} \right) \cup \right. \\
&\quad \left. \cup \partial \left(I_i^{n_0+(p_1+1)n_1} \times \dots \times I_1^{(p_i+1)n_i} \right) \right).
\end{aligned}$$

Using the Mayer-Vietoris Inequality (Lemma 1.5.23), this is bounded by

$$\sum_{1 \leq k_i \leq p_i+1} \sum_{\hat{J}_i^i \subset \{1, \dots, p_i+1\}, |\hat{J}_i^i|=k_i} b_{t_i-q_i-k_i} \left(\prod_{i, \hat{J}_i^i}^i \left(\prod_{i-1, J_{i-1}^1}^{1, i} \left(\bigcap_{j \in K_{i-1}} \overline{((X_i)_{\delta_j, \varepsilon_j})} \right) \cap \right. \right. \\ \left. \left. \cap \bigcap_{2 \leq r \leq i-1} \prod_{i-1, J_{i-1}^r}^{r, i} \overline{B_i^r \cap B_i^i} \right) \cup \partial \left(I_i^{n_0+(p_1+1)n_1} \times \dots \times I_1^{(p_i+1)n_i} \right) \right),$$

and, by Lemma 4.6.2, the summand is bounded by

$$b_{t_i-q_i-k_i} \left(\prod_{i, J_{i-1}^1 \times \hat{J}_i^i}^1 \left(\bigcap_{j \in K_{i-1}} \overline{((X_i)_{\delta_j, \varepsilon_j})} \right) \cap \bigcap_{2 \leq r \leq i} \prod_{i, J_{i-1}^r \times \hat{J}_i^i}^r \overline{B_i^r} \cup \right. \\ \left. \cup \partial \left(I_i^{n_0+(p_1+1)n_1} \times \dots \times I_1^{(p_i+1)n_i} \right) \right),$$

where $J_{i-1}^i = \{1\}$. Using the Mayer-Vietoris Inequality (Lemma 1.5.23), the last expression does not exceed

$$\sum_{1 \leq s_i \leq q_i+k_i+1+|K_{i-1}|} \sum_{J_i^1 \subset J_{i-1}^1 \times \hat{J}_i^i, \dots, J_i^i \subset J_{i-1}^i \times \hat{J}_i^i, K_i \subset K_{i-1}, |J_i^1|+\dots+|J_i^i|+|K_i|=s_i} \\ b_{t_i-q_i-k_i+s_i-1} \left(\bigsqcup_{i, J_i^1}^1 \left(\bigcup_{j \in K_i} \overline{((X_i)_{\delta_j, \varepsilon_j})} \right) \cup \bigcup_{2 \leq r \leq i} \bigsqcup_{i, J_i^r}^r \overline{B_i^r} \right. \\ \left. \cup \partial \left(I_i^{n_0+(p_1+1)n_1} \times \dots \times I_1^{(p_i+1)n_i} \right) \right).$$

We have $X_i = cp_{i+1}(X_{i+1})$, and by Lemma 4.9.3 $cp_{i+1}(X_{i+1})$ is represented by $cp((X_{i+1})_{\delta, \varepsilon})$,

so the summand of the previous expression is equal to

$$\begin{aligned}
& b_{t_i - q_i - k_i + s_i - 1} \left(\bigsqcup_{i, J_i^1}^1 \left(\bigcup_{j \in K_i} \overline{(cp_{i+1}((X_{i+1})_{\delta_j, \varepsilon_j}))} \right) \cup \right. \\
& \quad \left. \bigcup_{2 \leq r \leq i} \bigsqcup_{i, J_i^r}^r \overline{B_i^r} \cup \partial \left(I_i^{n_0 + (p_1 + 1)n_1} \times \dots \times I_1^{(p_i + 1)n_i} \right) \right) \\
&= b_{t_i - q_i - k_i + s_i - 1} \left(\bigsqcup_{i, J_i^1}^1 \left(\bigcup_{j \in K_i} \left(\pi_{i+1}(\overline{(X_{i+1})_{\delta_j, \varepsilon_j}}) \right) \right) \cup \right. \\
& \quad \left. \bigcup_{2 \leq r \leq i} \bigsqcup_{i, J_i^r}^r \overline{B_i^r} \cup \partial \left(I_i^{n_0 + (p_1 + 1)n_1} \times \dots \times I_1^{(p_i + 1)n_i} \right) \right) \\
&= b_{t_i - q_i - k_i + s_i - 1} \left(\bigsqcup_{i, J_i^1}^1 \bigcup_{j \in K_i} \left(\pi_{i+1}(\overline{(X_{i+1})_{\delta_j, \varepsilon_j}}) \right) \cup \right. \\
& \quad \left. \bigcup_{2 \leq r \leq i} \bigsqcup_{i, J_i^r}^r \pi_{i+1, r-1}(B_{i+1}^r) \cup \pi_{i+1, i} \left(\partial \left(I_i^{n_0 + (p_1 + 1)n_1} \times \dots \times I_1^{(p_i + 1)n_i} \right) \times I_1^{n_i + 1} \right) \right),
\end{aligned}$$

and bringing the projection to the front, this is equal to

$$b_{t_i - q_i - k_i + s_i - 1} \left(\pi_{i+1, i} \left(\bigsqcup_{i, J_i^1}^{1, i+1} \bigcup_{j \in K_i} \left(\overline{(X_{i+1})_{\delta_j, \varepsilon_j}} \right) \cup \bigcup_{2 \leq r \leq i} \bigsqcup_{i, J_i^r}^{r, i+1} B_{i+1}^r \cup B_{i+1}^{i+1} \right) \right).$$

Finally, from Lemma 4.7.1, this is bounded by

$$\sum_{p_{i+1} + q_{i+1} = t_i - q_i - k_i + s_i - 1} b_{q_{i+1}} \left(\prod_{i+1}^{i+1} \left(\bigsqcup_{i, J_i^1}^{1, i+1} \bigcup_{j \in K_i} \left(\overline{(X_{i+1})_{\delta_j, \varepsilon_j}} \right) \cup \bigcup_{2 \leq r \leq i} \bigsqcup_{i, J_i^r}^{r, i+1} B_{i+1}^r \cup B_{i+1}^{i+1} \right) \right).$$

The case when i is even follows directly by replacing $\bigcup_{j \in K_{i-1}} (X_i)_{\delta_j, \varepsilon_j}$ with $\bigcup_{j \in K_{i-1}} \overline{(X_i)_{\delta_j, \varepsilon_j}}$ at the start of the proof, immediately changing to

$$\bigcap_{j \in K_{i-1}} (X_i)_{\delta_j, \varepsilon_j}$$

in the next line and carrying this through to the stage where projections are introduced, replacing $\bigcup_{j \in K_i} (X_i)_{\delta_j, \varepsilon_j}$ with $\bigcup_{j \in K_i} \pi_{i+1}((X_{i+1})_{\delta_j, \varepsilon_j})$, and finishing with an expression involving $\bigcup_{j \in K_i} (X_{i+1})_{\delta_j, \varepsilon_j}$. \square

4.10 Upper Bounds

We have succeeded in finding an expression to bound each Betti number of X , in terms of some quantifier-free formula involving a set we can bound using the tools in the above chapters. It now remains to explicitly deduce a formula using the functions supplied by the complexity definition.

Theorem 4.9.8 tells us that $b_{q_0}(X)$ is bounded by:

$$\begin{aligned}
 & \sum_{p_1+q_1=q_0} \sum_{\hat{J}_1^1 \subset J_1^1} \sum_{J_2^2 \subset \hat{J}_1^1} \sum_{K_1 \subset \{1, \dots, m\}} \\
 & \sum_{p_2+q_2=t_1-q_1-|\hat{J}_1^1|-|K_1|-|J_2^2|-1} \sum_{1 \leq k_2 \leq p_2+1} \sum_{\hat{J}_2^2 \subset \{1, \dots, p_2+1\}, |\hat{J}_2^2|=k_2} \\
 & \sum_{1 \leq s_2 \leq q_2+k_2+1+|K_1|} \sum_{J_2^1 \subset J_1^1 \times \hat{J}_2^2, J_2^2 \subset J_1^2 \times \hat{J}_2^2, |J_2^1|+|J_2^2|=s_2} \sum_{p_3+q_3=t_2-q_2-k_2+s_2-1} \dots \\
 & \dots \sum_{1 \leq k_{i-1} \leq p_{i-1}+1} \sum_{\hat{J}_{i-1}^{i-1} \subset \{1, \dots, p_{i-1}+1\}, |\hat{J}_{i-1}^{i-1}|=k_{i-1}} \sum_{1 \leq s_{i-1} \leq q_{i-1}+k_{i-1}+1+|K_{i-2}|} \\
 & \sum_{J_{i-1}^1 \subset J_{i-2}^1 \times \hat{J}_{i-1}^{i-1}, \dots, J_{i-1}^{i-1} \subset J_{i-2}^{i-1} \times \hat{J}_{i-1}^{i-1}, K_{i-1} \subset K_{i-2}, |J_{i-1}^1| + \dots + |J_{i-1}^{i-1}| + |K_{i-1}| = s_{i-1}} \\
 & \sum_{p_i+q_i=t_{i-1}-q_{i-1}-k_{i-1}+s_{i-1}-1} \\
 & b_{q_i} \left(\prod_i \left(\bigcup_{i-1, J_{i-1}^1}^{1, i} \left(\bigcup_{j \in K_{i-1}} \overbrace{(X_i)_{\delta_j, \varepsilon_j}} \right) \cup \bigcup_{2 \leq r \leq i-1} \bigcup_{i-1, J_{i-1}^r}^{r, i} B_i^r \cup B_i^i \right) \right).
 \end{aligned}$$

We start by bounding the Betti numbers of the summand. To do this we prove the following lemma, which allows us to use the results in the previous sections.

Lemma 4.10.1. *Let X_ν be defined by a quantifier-free Boolean formula, consisting of s atoms of the type $f_i > 0$ or $f_i = 0$. Let t_ν be defined as in Definition 4.6.5. Then the set*

$$\prod_\nu \left(\bigcup_{\nu-1, J_{\nu-1}^1}^{1, \nu} \left(\bigcup_{j \in K_{\nu-1}} \overbrace{(X_\nu)_{\delta_j, \varepsilon_j}} \right) \cup \bigcup_{2 \leq r \leq \nu-1} \bigcup_{\nu-1, J_{\nu-1}^r}^{r, \nu} B_\nu^r \cup B_\nu^\nu \right) \subset \mathbb{R}^{t_\nu}$$

is defined by a quantifier-free Boolean formula with no negations, having at most

$$\begin{aligned}
 & \left((2t_{\nu-2} + 1) 4s(n_0 + \dots + n_\nu) + 2t_{\nu-1} + 2n_\nu + \left((4t_{\nu-1} + 2(n_\nu + \dots + n_\nu)) \cdot \right. \right. \\
 & \left. \left. \cdot (2t_{\nu-2} + 1) + 2t_{\nu-1} + 2n_\nu \right) (\nu - 2) \right) (t_{\nu-1} + 1) \leq st_{\nu-1}^{O(1)}
 \end{aligned}$$

atoms.

Proof. Our method consists of simple counting and calculation.

- The set $(X_\nu)_{\delta_j, \varepsilon_j}$ is defined by a Boolean formula with no negations consisting of $4s$ non-strict inequalities of the type $f + \delta_j$, $f - \delta_j$, $f + \varepsilon_j$, and $f - \varepsilon_j$.
- The set $\overbrace{(X_\nu)_{\delta_j, \varepsilon_j}}$ is either the same as the above, or its negation, which is a Boolean formula with no negations consisting of $4s$ strict inequalities. Either way, the set is defined by only non-strict or only strict inequalities, and no negations.
- We have $\{1, \dots, m\} \supset K_1 \supset \dots \supset K_\nu$, so $|K_i| \leq m$ for all i . We define m when constructing T , and from Theorem 4.5.7 we know if m is at least the dimension of the space, then T is a ‘good’ approximation. The maximum dimension we work in is $n_0 + \dots + n_\nu$. Therefore

$$\bigcup_{j \in K_{\nu-1}} \overbrace{(X_\nu)_{\delta_j, \varepsilon_j}}$$

is a set defined by a Boolean formula with no negations consisting of at most $4s(n_0 + \dots + n_\nu)$ only strict or only non-strict inequalities.

- The set

$$\bigsqcup_{\nu-1, J_{\nu-1}^1}^{1, \nu} \left(\bigcup_{j \in K_{\nu-1}} \overbrace{(X_\nu)_{\delta_j, \varepsilon_j}} \right) \subset \mathbb{R}^{t_{\nu-1} + n_\nu}$$

is defined by a Boolean formula with no negations having $|J_{\nu-1}^1|4s(n_0 + \dots + n_\nu) \leq s_{\nu-1}4s(n_0 + \dots + n_\nu) \leq (2t_{\nu-2} + 1)4s(n_0 + \dots + n_\nu)$ atoms, and at most $2t_{\nu-1} + 2n_\nu$ linear atoms (defining $I_1^{t_{\nu-1} + n_\nu}$).

- For any $2 \leq r \leq \nu$, the set $B_\nu^r \subset \mathbb{R}^{t_{r-1} + n_r}$ is defined by a Boolean formula with no negations having $4t_{r-1} + 2n_r$ linear atomic inequalities.
- Therefore, all sets of the kind B_j^r for $j \geq r$ are defined by Boolean formulae with no negations having $4t_{r-1} + 2(n_r + \dots + n_j)$ linear inequalities. In particular the set $B_\nu^r \subset \mathbb{R}^{t_{r-1} + n_r + \dots + n_\nu}$ is defined by $4t_{r-1} + 2(n_r + \dots + n_\nu)$ linear atomic inequalities.
- For any $2 \leq r \leq \nu - 1$, the set $\bigsqcup_{\nu-1, J_{\nu-1}^r}^{r, \nu} B_\nu^r \subset \mathbb{R}^{t_{\nu-1} + n_\nu}$ is defined by a formula with no negations having at most

$$\begin{aligned} & (4t_{r-1} + 2(n_r + \dots + n_\nu))|J_{\nu-1}^r| + 2t_{\nu-1} + 2n_\nu \\ & \leq (4t_{r-1} + 2(n_r + \dots + n_\nu))s_{\nu-1} + 2t_{\nu-1} + 2n_\nu \\ & \leq (4t_{r-1} + 2(n_r + \dots + n_\nu))(2t_{\nu-2} + 1) + 2t_{\nu-1} + 2n_\nu \end{aligned}$$

linear atoms.

- The set $\bigcup_{2 \leq r \leq \nu-1} \bigsqcup_{\nu-1, J_{\nu-1}^r}^{r, \nu} B_\nu^r \subset \mathbb{R}^{t_{\nu-1} + n_\nu}$ is therefore defined by a Boolean formula with no negations having at most

$$((4t_{r-1} + 2(n_r + \dots + n_\nu))(2t_{\nu-2} + 1) + 2t_{\nu-1} + 2n_\nu)(\nu - 2)$$

atoms.

- Therefore, the set

$$\prod_{\nu}^{\nu} \left(\bigsqcup_{\nu-1, J_{\nu-1}^1}^{1, \nu} \left(\bigcup_{j \in K_{\nu-1}} \overbrace{(X_{\nu})_{\delta_j, \varepsilon_j}} \right) \cup \bigcup_{2 \leq r \leq \nu-1} \bigsqcup_{\nu-1, J_{\nu-1}^r}^{r, \nu} B_{\nu}^r \cup B_{\nu}^{\nu} \right)$$

is defined by a Boolean formula with no negations having at most

$$\left((2t_{\nu-2} + 1) 4s(n_0 + \dots + n_{\nu}) + 2t_{\nu-1} + 2n_{\nu} + ((4t_{r-1} + 2(n_r + \dots + n_{\nu})) \cdot (2t_{\nu-2} + 1) + 2t_{\nu-1} + 2n_{\nu})(\nu - 2) \right) (t_{\nu-1} + 1) \leq st_{\nu-1}^{O(1)}$$

atoms.

□

We can now use Corollary 3.4.14, along with our particular defined complexity, to find an upper bound for

$$\prod_{\nu}^{\nu} \left(\bigsqcup_{\nu-1, J_{\nu-1}^1}^{1, \nu} \left(\bigcup_{j \in K_{\nu-1}} \overbrace{(X_{\nu})_{\delta_j, \varepsilon_j}} \right) \cup \bigcup_{2 \leq r \leq \nu-1} \bigsqcup_{\nu-1, J_{\nu-1}^r}^{r, \nu} B_{\nu}^r \cup B_{\nu}^{\nu} \right).$$

We now count the number of times this expression is added. We use the following notation, taken from [11]:

Definition 4.10.2. Let $f, g, h : \mathbb{N}^l \rightarrow \mathbb{N}$ be three functions. The expression $f \leq g^{O(h)}$ means that there exists $c \in \mathbb{N}$ such that $f \leq g^{ch}$ everywhere on \mathbb{N}^l .

Lemma 4.10.3. *The number of additive terms in (4.9.8) does not exceed*

$$2^{O(i^2(2(n_0 + \dots + n_{\nu}))^i n_0 n_1 \dots n_{i-2})}.$$

Proof. Firstly, we have to count the number of terms introduced by

$$\sum_{\hat{J}_1^1 \subset J_1^1} \sum_{J_2^2 \subset \hat{J}_1^1} \sum_{K_1 \subset \{1, \dots, m\}}.$$

We have $J_1^1 = \{1, \dots, p_1 + 1\}$, so the number of terms in the first summation is bounded by 2^{p_1+1} , as is that in the second summation, and the third summation has no more than 2^m terms.

Next we can partition the remaining terms into $i - 1$ groups of the kind

$$\sum_{1 \leq k_j \leq p_j + 1} \sum_{\hat{J}_j^j \subset \{1, \dots, p_j + 1\}, |\hat{J}_j^j| = k_j}$$

$$\begin{aligned}
& \sum_{1 \leq s_j \leq q_j + k_j + 1 + |K_{i-1}|} \\
& \sum_{J_j^1 \subset J_{j-1}^1 \times \hat{J}_j^j, \dots, J_j^j \subset J_{j-1}^j \times \hat{J}_j^j, K_j \subset K_{j-1}, |J_j^1| + \dots + |J_j^j| + |K_j| = s_j} \\
& \sum_{p_{j+1} + q_{j+1} = t_j - q_j - k_j + s_j - 1},
\end{aligned}$$

where $1 \leq j \leq i - 1$.

The number of terms in

$$\sum_{1 \leq k_j \leq p_j + 1} \sum_{\hat{J}_j^j \subset \{1, \dots, p_j + 1\}, |\hat{J}_j^j| = k_j}$$

is 2^{p_j+1} .

The number of terms in

$$\begin{aligned}
& \sum_{1 \leq s_j \leq q_j + k_j + 1 + |K_{i-1}|} \\
& \sum_{J_j^1 \subset J_{j-1}^1 \times \hat{J}_j^j, \dots, J_j^j \subset J_{j-1}^j \times \hat{J}_j^j, K_j \subset K_{j-1}, |J_j^1| + \dots + |J_j^j| + |K_j| = s_j}
\end{aligned}$$

does not exceed $2^{(j+1)(q_j + k_j + 1 + |K_{i-1}|)} \leq 2^{(j+1)(q_j + k_j + 1 + n_0 + \dots + n_\nu)}$.

The number of terms in

$$\sum_{p_{j+1} + q_{j+1} = t_j - q_j - k_j + s_j - 1}$$

does not exceed $t_j + 1$.

Therefore, it follows that the total number of terms in the j th group does not exceed

$$2^{p_j+1+(j+1)(q_j+k_j+1+n_0+\dots+n_\nu)}(t_j + 1).$$

We have $s_j \leq q_j + k_j + 1 + |K_{j-1}|$, so $t_j - q_j - k_j + s_j - 1 \leq t_j + |K_{j-1}|$, and therefore $p_{j+1} + q_{j+1} \leq t_j + |K_{j-1}|$. Hence we have

$$\begin{aligned}
& 2^{p_j+1+(j+1)(q_j+k_j+1+n_0+\dots+n_\nu)}(t_j + 1) \\
& \leq 2^{p_j+1+(j+1)(q_j+p_j+2+n_0+\dots+n_\nu)}(t_j + 1) \\
& \leq 2^{(j+1)(p_j+1+q_j+p_j+2+n_0+\dots+n_\nu)}(t_j + 1) \\
& \leq 2^{(j+1)(2p_j+q_j+3+n_0+\dots+n_\nu)}(t_j + 1) \\
& \leq 2^{O(j(t_{j-1}+n_0+\dots+n_\nu))}.
\end{aligned}$$

Since $t_j = n_0 + n_1(p_1 + 1) + \dots + n_j(p_j + 1)$, $p_l \leq t_{l-1} + |K|$, and therefore $t_j \leq (2|K|)^j n_0 n_1 \dots n_j + 1$ (proof of this follows easily using induction, see Lemma 4.10.5 below), the number of terms in the j th group does not exceed

$$2^{O(j(2|K|)^j n_0 n_1 \dots n_{j-1})} \leq 2^{O(j(2(n_0+\dots+n_\nu))^j n_0 n_1 \dots n_{j-1})}.$$

It follows that the total number of terms in (4.9.8) does not exceed

$$2^{O(i^2(2(n_0+\dots+n_\nu))^i n_0 n_1 \dots n_{i-2})}.$$

□

Remark 4.10.4. Thanks to James Davenport for suggesting this improvement during the viva. In the preceding we counted each of the Σ separately, for example for

$$\sum_{\hat{J}_1^1 \subset J_1^1} \sum_{J_2^2 \subset \hat{J}_1^1}$$

we bounded each of the summations by 2^{p_1+1} and get a total of 4^{p_1+1} . We could, however rewrite this as

$$\sum_{J_2^2 \subset \hat{J}_1^1 \subset J_1^1}$$

and bound this by 3^{p_1+1} , resulting in a slightly better bound. Similar techniques could be used throughout. This does not produce any difference at the level of $2^{O(\dots)}$, but may be useful in future work.

Lemma 4.10.5.

$$t_j \leq (2|K|)^j n_0 n_1 \dots n_j + 1$$

Proof. By induction. Firstly, we have $t_j = n_0 + n_1(p_1 + 1) + \dots + n_j(p_j + 1)$ and $p_l \leq t_{l-1} + |K|$.

- For $j = 0$, $t_0 = n_0$

- For $j = 1$,

- If $n_0 > 1$ or $n_1 > 1$

$$\begin{aligned} t_1 &= n_0 + n_1(p_1 + 1) \\ &\leq n_0 + n_1(n_0 + |K| + 1) \\ &= n_0 + n_1 + n_0 n_1 + n_1 |K| \\ &\leq 2n_0 n_1 + n_1 |K|, \text{ since } n_0 \text{ and } n_1 \text{ are positive integers and not both } 1 \\ &\leq 2|K|n_0 n_1, \text{ as } |K| \geq 2 \end{aligned}$$

- If $n_0 = n_1 = 1$ we have

$$\begin{aligned} t_1 &= n_0 + n_1(p_1 + 1) \\ &\leq 1 + 1(1 + |K| + 1) \\ &= |K| + 3 \text{ and as } |K| \geq 2 \\ &\leq 2|K| + 1 \end{aligned}$$

- Assume $t_j \leq (2|K|)^j n_0 n_1 \dots n_j$. Then

$$\begin{aligned}
t_{j+1} &= t_j + n_{j+1}(p_{j+1} + 1) \\
&\leq t_j + n_{j+1}(t_j + |K| + 1) \\
&\leq (2|K|)^j n_0 n_1 \dots n_j + (2|K|)^j n_0 n_1 \dots n_j n_{j+1} + n_{j+1}|K| + n_{j+1} \\
&\leq 2^{j+1}|K|^j n_0 n_1 \dots n_j n_{j+1} + n_{j+1}|K| \\
&\leq (2|K|)^{j+1} n_0 n_1 \dots n_j n_{j+1}
\end{aligned}$$

□

We now know how to calculate the Betti numbers of the summand, and we know how many times we are summing, so we only need to use a result from the previous sections to produce a concrete bound.

We will use Corollary 3.4.14, (which is equivalent to the bound $O(s^2 d)^n$ in the basic polynomial, degree case).

Definition 4.10.6. For a set of functions $F = \{f_1, \dots, f_s\}$ from $H \rightarrow \mathbb{R}$, where $H \subset \mathbb{R}^n$ define the function

$$\Omega(F) = \max_{1 \leq i \leq s} \left(\frac{\gamma(n, c(f_i^2 + |x|^2))}{2} \right).$$

Note the following is really a Corollary of Theorem 4.9.8, but is called a Theorem because it is one of the most significant results of this thesis.

Theorem 4.10.7 (Main Result B). *Let C denote the maximum complexity of the atoms in*

$$Z = \prod_{\nu} \left(\bigsqcup_{\nu-1, J_{\nu-1}^1}^{1, \nu} \left(\bigcup_{j \in K_{\nu-1}} \overbrace{(X_{\nu})_{\delta_j, \varepsilon_j}} \right) \cup \bigcup_{2 \leq r \leq \nu-1} \bigsqcup_{\nu-1, J_{\nu-1}^r}^{r, \nu} B_{\nu}^r \cup B_{\nu}^{\nu} \right).$$

Let F be a set containing elements $g, |x|^2$, where g is any function of complexity C with domain $\mathbb{R}^{t_{\nu}}$. Let $u_j = n_0 + n_1 + \dots + n_j$, and $w_j = n_0 n_1 \dots n_j$. Then:

$$b_{q_0}(X) \leq (2^{\nu^2} u_{\nu}^{\nu} s w_{\nu-1})^{O((2u_{\nu})^{\nu} w_{\nu})} \Omega(F).$$

Proof. We take Theorem 4.9.8 in the case $i = \nu$. Firstly, we directly apply the bound given in 3.4.14 to the set $Z \subset \mathbb{R}^{t_{\nu}}$, which is defined by $st_{\nu-1}^{O(1)}$ atoms, and get

$$b_{q_0}(Z) \leq O((st_{\nu-1}^{O(1)})^{2t_{\nu}} \Omega(F))$$

From Lemma 4.10.3 we know that we are adding this together

$$2^{O(\nu^2(2(n_0 + \dots + n_{\nu}))^{\nu} n_0 n_1 \dots n_{\nu-2})}$$

times, so we have

$$\begin{aligned} b_{q_0}(X) &\leq 2^{O(\nu^2(2(n_0+\dots+n_\nu))^\nu n_0 n_1 \dots n_{\nu-2})} O((st_{\nu-1}^{O(1)})^{2t_\nu} \Omega(F)) \\ &\leq 2^{O(\nu^2(2(n_0+\dots+n_\nu))^\nu n_0 n_1 \dots n_{\nu-2})} O((st_{\nu-1})^{O(t_\nu)} \Omega(F)) \end{aligned}$$

We have $t_j \leq (2|K|)^j n_0 n_1 \dots n_j \leq (2(n_0 + \dots + n_\nu))^j n_0 n_1 \dots n_j$, so

$$\begin{aligned} b_{q_0}(X) &\leq 2^{O(\nu^2(2(n_0+\dots+n_\nu))^\nu n_0 n_1 \dots n_{\nu-2})} \\ &\cdot O((s(2(n_0 + \dots + n_\nu))^{\nu-1} n_0 n_1 \dots n_{\nu-1})^{O((2(n_0+\dots+n_\nu))^\nu n_0 n_1 \dots n_\nu)} \Omega(F)) \end{aligned}$$

And substituting $u_j = n_0 + n_1 + \dots + n_j$, $w_j = n_0 n_1 \dots n_j$ gives

$$\begin{aligned} b_{q_0}(X) &\leq 2^{O(\nu^2(2u_\nu)^\nu w_{\nu-2})} O((s(2u_\nu)^{\nu-1} w_{\nu-1})^{O((2u_\nu)^\nu w_\nu)} \Omega(F)) \\ &\leq (2^{\nu^2} u_\nu^\nu s w_{\nu-1})^{O((2u_\nu)^\nu w_\nu)} \Omega(F). \end{aligned}$$

□

In the preceding result, and the following examples, we see a u_ν^ν factor appearing in a number of places. This is the consequence of not restricting to the case of X_ν being open or closed, and thus needing to use the T construction to approximate by a compact set. To construct T we take the union of m sets, and we use $n_0 + \dots + n_\nu = u_\nu$ to bound m , which results in this extra factor. A possible improvement would be to find a tighter bound for m .

Example 4.10.8 (The Degree of a Polynomial). *If the original polynomials have maximum degree d , then the polynomials defining Z will also have maximum degree d , and $\Omega(F) = O(d^{t_\nu}) \leq O(d^{(2(n_0+\dots+n_\nu))^\nu n_0 n_1 \dots n_\nu})$, so*

$$b_{q_0}(X) \leq (2^{\nu^2} u_\nu^\nu d s w_{\nu-1})^{O((2u_\nu)^\nu w_\nu)}.$$

Note the difference to the bound of

$$(2^{\nu^2} d s w_{\nu-1})^{O(2^\nu w_\nu)}.$$

in section 8.1 of [12] for the case where X_ν is open or closed.

Example 4.10.9 (Pfaffian Functions). *If the functions defining X_ν are Pfaffian functions of order r , degree (α, β) , then the functions defining Z are Pfaffian functions of degree (α, β) and*

order at most $r(2t_{\nu-2} + 1)(t_{\nu-1} + 1) = O(rt_{\nu-2}t_{\nu-1})$. We then have

$$\begin{aligned}
\Omega(F) &= 2^{O(rt_{\nu-2}t_{\nu-1})(O(rt_{\nu-2}t_{\nu-1})-1)/2} \beta(\alpha + 2\beta - 1)^{t_{\nu}-1} \\
&\quad \cdot (\min\{t_{\nu}, O(rt_{\nu-2}t_{\nu-1})\} \alpha + 2t_{\nu}\beta + (t_{\nu} - 1)\alpha - 2t_{\nu} + 2)^{O(rt_{\nu-2}t_{\nu-1})} \\
&\leq 2^{O(rt_{\nu-2}t_{\nu-1})^2} \beta(\alpha + 2\beta - 1)^{t_{\nu}-1} \\
&\quad \cdot (\min\{t_{\nu}, O(rt_{\nu-2}t_{\nu-1})\} \alpha + 2t_{\nu}\beta + (t_{\nu} - 1)\alpha - 2t_{\nu} + 2)^{O(rt_{\nu-2}t_{\nu-1})} \\
&\leq 2^{O(rt_{\nu-2}t_{\nu-1})^2} (\alpha + \beta)^{O(t_{\nu})} \\
&\quad \cdot (\min\{t_{\nu}, O(rt_{\nu-2}t_{\nu-1})\} \alpha + 2t_{\nu}\beta + (t_{\nu} - 1)\alpha - 2t_{\nu} + 2)^{O(rt_{\nu-2}t_{\nu-1})} \\
&\leq 2^{O(rt_{\nu-2}t_{\nu-1})^2} (\alpha + \beta)^{O(t_{\nu})} (t_{\nu}(\alpha + \beta))^{O(rt_{\nu-2}t_{\nu-1})}.
\end{aligned}$$

We have $t_j \leq (2(n_0 + \dots + n_{\nu}))^j n_0 n_1 \dots n_j = (2u_{\nu})^j w_j$, so

$$\begin{aligned}
\Omega(F) &\leq 2^{O(r(2u_{\nu})^{\nu-2} w_{\nu-2} (2u_{\nu})^{\nu-1} w_{\nu-1})^2} (\alpha + \beta)^{O((2u_{\nu})^{\nu} w_{\nu})} \\
&\quad ((2u_{\nu})^{\nu} w_{\nu} (\alpha + \beta))^{O(r(2u_{\nu})^{\nu-2} w_{\nu-2} (2u_{\nu})^{\nu-1} w_{\nu-1})} \\
&\leq 2^{O(r^2 (2u_{\nu})^{4\nu} w_{\nu-2}^2 w_{\nu-1}^2)} (\alpha + \beta)^{O((2u_{\nu})^{\nu} w_{\nu})} ((2u_{\nu})^{\nu} w_{\nu} (\alpha + \beta))^{O(r(2u_{\nu})^{2\nu} w_{\nu-2} w_{\nu-1})}.
\end{aligned}$$

Now we have

$$\begin{aligned}
b_{q_0}(X) &\leq (2^{\nu^2} u_{\nu}^{\nu} s w_{\nu-1})^{O((2u_{\nu})^{\nu} w_{\nu})} 2^{O(r^2 (2u_{\nu})^{4\nu} w_{\nu-2}^2 w_{\nu-1}^2)} (\alpha + \beta)^{O((2u_{\nu})^{\nu} w_{\nu})} \\
&\quad ((2u_{\nu})^{\nu} w_{\nu} (\alpha + \beta))^{O(r(2u_{\nu})^{2\nu} w_{\nu-2} w_{\nu-1})} \\
&\leq s^{O((2u_{\nu})^{\nu} w_{\nu})} (w_{\nu-1}^{O((2u_{\nu})^{\nu} w_{\nu})}) (2^{\nu^2} u_{\nu}^{\nu})^{O((2u_{\nu})^{\nu} w_{\nu})} 2^{O(r^2 (2u_{\nu})^{4\nu} w_{\nu-2}^2 w_{\nu-1}^2)} (\alpha + \beta)^{O((2u_{\nu})^{\nu} w_{\nu})} \\
&\quad ((2u_{\nu})^{\nu} w_{\nu} (\alpha + \beta))^{O(r(2u_{\nu})^{2\nu} w_{\nu-2} w_{\nu-1})} \\
&\leq s^{O((2u_{\nu})^{\nu} w_{\nu})} (2^{\nu^2} u_{\nu}^{\nu})^{O((2u_{\nu})^{\nu} w_{\nu})} 2^{O(r^2 (2u_{\nu})^{4\nu} w_{\nu-2}^2 w_{\nu-1}^2)} (w_{\nu-1} (\alpha + \beta))^{O((2u_{\nu})^{\nu} w_{\nu})} \\
&\quad ((2u_{\nu})^{\nu} w_{\nu} (\alpha + \beta))^{O(r(2u_{\nu})^{2\nu} w_{\nu-2} w_{\nu-1})} \\
&\leq s^{O((2u_{\nu})^{\nu} w_{\nu})} (2^{\nu^2} u_{\nu}^{\nu})^{O((2u_{\nu})^{\nu} w_{\nu})} 2^{O(r^2 (2u_{\nu})^{4\nu} w_{\nu-2}^2 w_{\nu-1}^2)} ((2u_{\nu})^{\nu})^{O(r(2u_{\nu})^{2\nu} w_{\nu-2} w_{\nu-1})} \\
&\quad (w_{\nu} (\alpha + \beta))^{O((2u_{\nu})^{\nu} w_{\nu} + r(2u_{\nu})^{2\nu} w_{\nu-2} w_{\nu-1})} \\
&\leq s^{O((2u_{\nu})^{\nu} w_{\nu})} (2^{\nu^2} u_{\nu}^{\nu})^{O((2u_{\nu})^{\nu} w_{\nu})} 2^{O(r^2 (2u_{\nu})^{4\nu} w_{\nu-2}^2 w_{\nu-1}^2)} ((2u_{\nu})^{\nu})^{O(r(2u_{\nu})^{2\nu} w_{\nu-2} w_{\nu-1})} \\
&\quad (w_{\nu} (\alpha + \beta))^{O((2u_{\nu})^{\nu} w_{\nu} + r(2u_{\nu})^{2\nu} w_{\nu-2} w_{\nu-1})} \\
&\leq s^{O((2u_{\nu})^{\nu} w_{\nu})} (2^{\nu^2} u_{\nu}^{\nu})^{O((2u_{\nu})^{\nu} w_{\nu})} (2u_{\nu})^{O(\nu(r(2u_{\nu})^{2\nu} w_{\nu-2} w_{\nu-1}) + r^2 (2u_{\nu})^{4\nu} w_{\nu-2}^2 w_{\nu-1}^2)} \\
&\quad (w_{\nu} (\alpha + \beta))^{O((2u_{\nu})^{\nu} w_{\nu} + r(2u_{\nu})^{2\nu} w_{\nu-2} w_{\nu-1})} \\
&\leq s^{O((2u_{\nu})^{\nu} w_{\nu})} (2^{\nu^2} u_{\nu}^{\nu})^{O((2u_{\nu})^{\nu} w_{\nu})} (2u_{\nu})^{O(r^2 (2u_{\nu})^{4\nu} w_{\nu-2}^2 w_{\nu-1}^2)} \\
&\quad (w_{\nu} (\alpha + \beta))^{O((2u_{\nu})^{\nu} w_{\nu} + r(2u_{\nu})^{2\nu} w_{\nu-2} w_{\nu-1})} \\
&\leq s^{O((2u_{\nu})^{\nu} w_{\nu})} (2^{\nu} u_{\nu})^{O(\nu(2u_{\nu})^{\nu} w_{\nu} + r^2 (2u_{\nu})^{4\nu} w_{\nu-2}^2 w_{\nu-1}^2)} \\
&\quad (w_{\nu} (\alpha + \beta))^{O((2u_{\nu})^{\nu} w_{\nu} + r(2u_{\nu})^{2\nu} w_{\nu-2} w_{\nu-1})}.
\end{aligned}$$

Compare this with the bound of

$$s^{O(2^\nu w_\nu)} 2^{O(\nu 2^\nu w_\nu + r^2 2^{4\nu} w_{\nu-2}^2 w_{\nu-1}^2)} (w_\nu(\alpha + \beta))^{O(2^\nu w_\nu + r 2^{2\nu} w_{\nu-2} w_{\nu-1})}$$

given in section 8.2 of [12] for the case where X_ν is open or closed.

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